Hyperarithmetical Properties of Relations on Abelian P-Groups and Orderings

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ABSTRACT.

An r.e. structure $\mathfrak{A}$ consists of given r.e. relations and recursive functions on a recursive universe. Given another relation R on the structure, we ask whether the structure can be renumbered to form a new (but classically isomorphic) r.e. structure on which the corresponding relation is not a Σ^0_α set (where α is a constructive ordinal). If this cannot be done, we say that R is *intrinsically* Σ^{0+}_α on $\mathfrak{A}$.

If R can be defined by a recursive infinitary formula of a certain kind, involving the given relations and functions of the structure, then we say R is *formally* Σ^{0+}_α. This definition guarantees that R is intrinsically Σ^{0+}_α.

In this thesis we show that

If certain conditions guaranteeing extra decidability are satisfied, then R is intrinsically Σ^{0+}_α *iff it is formally* Σ^{0+}_α.

The required decidability conditions are established for certain linear orderings and reduced abelian p-groups, and this result is applied to various relations in these cases.

Related questions are discussed in the case of the reduced abelian p-groups, and the thesis concludes with an example showing a difference between r.e. and recursive structures.

DECLARATION.

This thesis has been composed by me, and contains no material which has been submitted for any degree or qualification and, to the best of my knowledge and belief, no material previously published or written by another person, except where due reference is made in the text.

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CHAPTER 0. INTRODUCTION.

Although for most of the history of mathematics mathematicians have taken a constructive or algorithmic approach to mathematical structures and their properties, in the last hundred and fifty years the fashion has been largely for non-constructive or existential methods. A small counter-current was provided by the intuitionists and people with related constructivist programmes (for example, Bishop [Bi] and Richman [Ri]). But only the development of recursion theory, and the realization that it is the appropriate way to study the theoretical limits of the capabilities of computers and computation, has led, in the last forty years, to the systematic study of mathematical structures from an effective point of view. Here, in contrast to an intuitionistic approach, the mathematics is classical; the effectiveness of a procedure is given by its representation as a recursive function.

The beginnings of this approach lie in papers by Fröhlich and Shepherdson [FSh] and Rabin [Ra]. A major development took place in 1975 when Metakides and Nerode [MN] applied Friedberg and Muchnik's priority method to algebraic constructions. Since then these methods have been applied to a wide variety of algebraic structures, and the field of "effective algebra", as it has come to be known, has developed rapidly. A wide range of papers on various aspects of effective algebra may be found in [Cr].

General results, not tied to particular algebraic structures, have only appeared more recently. They began with work by Nurtazin

[Nt], Goncharov [G1], and Ash and Nerode [AN]. Other related work may be found in [LSc] and [Mi].

In a recursive structure certain relations and functions are recursive by the definition of the structure. Others may or may not be recursive, depending on the particular copy of the structure being considered. For example, in a recursive linear ordering the underlying set and the order relation $<$ are recursive by definition, but the successor relation might not be recursive. In [AN] this situation was studied in general and a condition was developed that, in the presence of certain decidability assumptions, determines exactly which new relations are r.e. in every recursive copy of the structure. This condition amounts to a correspondence between the syntactic and semantic properties of the relations. A relation is intrinsically r.e. (that is, the corresponding relation is r.e. in every recursive copy of the structure) iff it is formally r.e. (that is, it is definable by an r.e. disjunction of existential formulae involving only the given relations and functions on the structure). The necessity of the extra decidability assumptions was demonstrated in [Ma].

In [B1] this was generalized from r.e. relations to Σ_2^0 relations, with a construction involving an infinite injury priority argument. In this thesis the main result of Chapter 2 is the generalization to Σ_α^0 relations (where α is a constructive ordinal) in the more general setting of r.e. structures. This uses the α-systems of Chris Ash in order to carry out an α^{th} level priority argument. The result for Σ_α^0 relations on recursive structures was reported in [B2], a copy of which is included as an appendix to this thesis. (Sections 3 to 6 of [B2] contain work from [B1].)

The structure of the thesis is as follows.

Chapter 1 deals with some preliminary matters, and sections 2.1 to 2.5 of Chapter 2 contains the definitions and lemmas leading up to the main result. These are all, except for 2.2.4, 2.2.5, 2.3.6, 2.4.4 and 2.4.5, (variants of) definitions and results that appear or are implicit in [AN], [A1], [A2] and [A4]. Sections 2.6 to 2.8 contain the main result and some extensions, including the deduction of the result for recursive structures from the result for r.e. structures.

Chapter 3 contains some applications of the main result to recursive well orderings and some other well behaved linear orderings. Most of the chapter is devoted to determining the back and forth relations $\leq_\gamma$ which are needed for the decidability assumptions. The results for well orderings were reported in [B2]. Other, earlier work on recursive linear orderings may be found in [Re] and [Mo].

The back and forth relations for reduced abelian p-groups, which constitute the second main result of the thesis, are determined for any recursive reduced abelian p-group with a recursive sequence of Ulm invariants in Chapter 4. This generalizes the result for $\leq_1$ given in [B1].

Various applications of these are made in Chapter 5. We consider whether various relations are intrinsically Σ^0_α, applying the results of Chapter 2. We also ask when a recursive reduced abelian p-group is Δ^0_α categorical, applying a result from [A3].

Finally we consider the construction of "pairs of groups", applying a result from [AKn]. Earlier related work on groups may be found in [BwE], [G2], [Li], [Rol] and [Sm].

In Chapter 6 we give a simple example involving a partial ordering which illustrates the difference between a structure considered as recursive and as r.e.

I would like to thank Dr. Chris Ash for his help, encouragement and advice, Prof. Julia Knight for stimulating conversations, and Dr. Ewan Klein for help with administrative matters. I also wish to thank Dr. Ash for permission to include my proof of his result, Proposition 3.2.3.

CHAPTER 1. PRELIMINARIES.

§1.1 Notation.

Infinitary formulae.

The formulae we will consider are the infinitary formulae from the appropriate $L_{\omega_1\omega}$, as defined in [Ke].

We use the symbol $\bigvee$ to denote an infinite disjunction, while $\vee$ denotes a finite disjunction. Similarly $\bigwedge$ and $\wedge$ denote respectively infinite and finite conjunctions.

For a structure $\mathfrak{A}$ and a set of sentences S in a language suitable for $\mathfrak{A}$, we denote the set of sentences in S satisfied by $\mathfrak{A}$ by $S(\mathfrak{A})$. We sometimes abuse language by saying that a formula is in S when it is logically equivalent to a formula in S.

Sequences.

We use bold faced letters $\mathbf{a}$ to denote configurations of the form $a_1, a_2, \ldots, a_n$ which may, depending on context, also be interpreted as the set $\{a_1, a_2, \ldots, a_n\}$ or the sequence $(a_1, a_2, \ldots, a_n)$. Thus $\mathbf{a}, \mathbf{b}$ may denote the concatenation of the sequences $\mathbf{a}$ and $\mathbf{b}$. Also $f(\mathbf{a})$ will denote $f(a_1), f(a_2), \ldots, f(a_n)$.

Recursion theoretic notation.

The value computed by the e^{th} Turing machine with oracle A and input x is denoted by $\varphi_e^A(x)$. So $\varphi_e^A(x)$ may be undefined. $\varphi_e^{A,s}(x)$ denotes the value (if any) computed in at most s steps.

Since we have an effective listing of Turing machines, $\{ (x,y,e,s) : \varphi_e^{A,s}(x) = y \}$ is recursive relative to the oracle A.

Also

$$\varphi_e^A(x) = y \longleftrightarrow \exists s\ (\varphi_e^{A,s}(x) = y)$$

so $\{ (x,y,e) : \varphi_e^A(x) = y \}$ is r.e. relative to A.

The e^{th} r.e. set, relative to oracle A, is

$$W_e^A = \{ x : \exists y\ (\varphi_e^A(x) = y) \}.$$

The notation $\langle\ \rangle$ will mean, according to context, a standard pairing, tripling or quadrupling function, or a canonical index (that is, a function which maps sequences of arbitrary finite length onto indices in $\mathbb{N}$).

In Chapters 4 and 5, however, $\langle \mathbf{a} \rangle$ will denote the group generated by $\mathbf{a}$.

For unexplained model theoretic terminology see [CKe], and for unexplained recursion theoretic terminology see [Ro] or [So].

§1.2 Ordinal Notations.

Kleene's system $\mathcal{O}$ of ordinal notations (See [Ro], §11.7, §11.8) provides effective notations for a denumerable initial segment of the ordinals, called the *constructive ordinals,* which coincides with the recursive ordinals. (α is a *recursive ordinal* if there is a recursive linear ordering of order type α.)

Definition 1.2: The system $\mathcal{O}$ is defined as follows. We define the function $|\,|_{\mathcal{O}}$ from a set of integers $D_{\mathcal{O}}$ (i.e. the notations) to a segment of the ordinal numbers ($|a|_{\mathcal{O}} = \alpha$ means that a is a notation for α) and the partial ordering $<_{\mathcal{O}}$ by induction on ordinals.

ϕ receives notation *1.*

Assume all ordinals $< \gamma$ have received their notations, and $<_{\mathcal{O}}$ has been defined on these notations.

(i) If $\gamma = \beta+1$, then for each x such that $|x|_{\mathcal{O}} = \beta$, γ receives 2^{β} as a notation; and the ordered pairs $(z,2^{x})$ are added to the relation $<_{\mathcal{O}}$ for all z for which either $z=x$ or (z,x) is already in $<_{\mathcal{O}}$.

(ii) If γ is a limit ordinal, then for each y such that $\{\varphi_{y}(n)\}$ are notations for an increasing sequence of ordinals γ_{n} with limit γ and such that for all $i<j$ $(\varphi_{y}(i),\varphi_{y}(j))$ is already in $<_{\mathcal{O}}$, γ receives 3.5^{y} as a notation; and the ordered pairs $(z,3.5^{y})$ are added to $<_{\mathcal{O}}$ for all z for which $(z,\varphi_{y}(n))$ is already in $<_{\mathcal{O}}$ for some n.

Clearly there are partial recursive functions which, given any notation $a \in D_{\mathcal{O}}$: (a) tell us whether $|a|_{\mathcal{O}}$ is ϕ, a successor or a limit ordinal; (b) if $|a|_{\mathcal{O}}$ is a successor, give us (a notation for) its predecessor; and (c) if $|a|_{\mathcal{O}}$ is a limit, give us (notations for) an increasing sequence of ordinals with limit $|a|_{\mathcal{O}}$.

If we consider a fixed constructive ordinal α, and a given notation a for α, then each ordinal $\beta \leq \alpha$ has a unique notation b such that $b <_{\mathcal{O}} a$. This will always be the situation considered in the sequel. Furthermore, in the interests of readability we will sometimes refer to ordinals rather than notations for ordinals.

The first ordinal not given a notation in $\mathcal{O}$ is denoted ω_1^{CK}. It is denumerable, but unreachable by any effective process.

By "recursive transfinite induction" we mean Kleene's method for obtaining a partial recursive function h on $\mathcal{O}$ by defining each $h(a)$ in terms of $\{ h(b) : b <_{\mathcal{O}} a \}$ and applying the Recursion Theorem. (See [Ro].)

§1.3 The Hyperarithmetical Hierarchy.

The arithmetical hierarchy is a classification of the complexity of sets definable from recursive sets using first order formulae. The hyperarithmetical hierarchy stands in an analogous relation to the infinitary formulae of $L_{\omega_1\omega}$. It is an extension of the classification of sets as Σ_n, Π_n or Δ_n, where $n \in \mathbb{N}$, to a classification as Σ_α, Π_α or Δ_α, where α is a constructive ordinal.

Definition 1.3: We associate sets with notations in $\mathcal{O}$ as follows.

$$H(1) = \phi$$

$$H(2^x) = (H(x))' \text{ for } x \in D_{\mathcal{O}}$$

(' denotes the recursion theoretic jump operation)

$$H(3.5^y) = \{ (u,v) : v <_{\mathcal{O}} 3.5^y \ \& \ u \in H(v) \} \quad \text{for } 3.5^y \in D_{\mathcal{O}}$$

Now for $\alpha = n+1 < \omega$, let $|a|_{\mathcal{O}} = n$ and define

$$\Sigma^0_\alpha = \Sigma^{H(a)}_1 = \{ W^{H(a)}_e : e \in \mathbb{N} \} \text{ and } \Pi^0_\alpha = \{ S : \bar{S} \in \Sigma^0_\alpha \}$$

(This is the arithmetical hierarchy.)

For $\alpha \geq \omega$, let $|a|_{\mathcal{O}} = \alpha$ and define

$$\Sigma^0_\alpha = \Sigma^{H(a)}_1 \text{ and } \Pi^0_\alpha = \{ S : \bar{S} \in \Sigma^0_\alpha \}$$

Finally let $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$.

This constitutes the hyperarithmetical hierarchy. For more information, see [Ro].

A Σ^0_α *index* for a Σ^0_α set S is a pair $\langle e,a\rangle$ where e is an r.e. index for S relative to the oracle $H(a)$. We write $S = W_{e,a}$.

Clearly, from a Σ^0_α index for S, we can effectively find a Σ^0_γ index for S, for $\gamma \geq \alpha$.

CHAPTER 2. INTRINSICALLY Σ^0_α RELATIONS.

§2.1 R.E. Structures.

Definition 2.1.1: An *r.e. structure* is a structure of the form

$$\mathfrak{A} = (A, \langle R_i\rangle_{i\in I}, \langle g_j\rangle_{j\in J}, \langle a_k\rangle_{k\in K})$$

for which A, I, J, K are recursive sets and, for suitable recursive functions μ, ν, each R_i is uniformly in i an r.e. $\mu(i)$-ary recursive relation on A, each g_j is uniformly in j a $\nu(j)$-ary recursive function on A and the function $k \longmapsto a_k$ from K to A is recursive. The *similarity type* of such a structure is (I, J, K, μ, ν) and the corresponding effective language is denoted by L.

It suffices to consider r.e. structures of the form $(A, \langle R_i\rangle_{i\in I})$ since the g_j and a_k may be represented by relations in the usual way. (A total function is recursive iff its graph is r.e. iff its graph is recursive.)

If the R_i are required to be uniformly recursive, then the above structure is a *recursive structure*. Again, it suffices to consider structures of the form $(A, \langle R_i\rangle_{i\in I})$. Note that results about r.e structures imply the corresponding results about recursive structures, since a recursive structure $(A, \langle R_i\rangle_{i\in I})$ corresponds to an r.e. structure $(A, \langle R_i\rangle_{i\in I}\langle \bar{R}_i\rangle_{i\in I})$, where $\bar{R}_i$ denotes the complement of R_i.

A recursive permutation of the set A in an r.e. structure $\mathfrak{A}$ obviously does not change the recursive properties of $\mathfrak{A}$. We view such a permutation as an isomorphism $f : \mathfrak{A} \cong \mathfrak{B}$ between r.e. structures, where $\mathfrak{B} = (A, \langle f(R_i)\rangle_{i\in I})$.

A non-recursive permutation may still yield an r.e. structure but now some of the recursive properties may be changed.

For example in a 'nice' copy of a recursive linear ordering of order type ω^2 the set of left limit points P would be recursive, even though this is not required by the definition of a recursive linear ordering. In [B1] it is shown that there is a recursive linear ordering of order type ω^2 in which P is not Σ^0_2.

In this example the set P is not intrinsically Σ^0_2.

Definition 2.1.2: A relation R on an r.e. structure $\mathfrak{A}$ is *intrinsically* Σ^{0+}_α if for every isomorphism $f : \mathfrak{A} \cong \mathfrak{B}$ between $\mathfrak{A}$ and another r.e. structure $\mathfrak{B}$, its image $f(R)$ forms a Σ^0_α set.

On a recursive structure we define intrinsically Σ^0_α relations in the corresponding way.

§2.2 Infinitary Formulae.

Following [A4], we define the Σ^+_α and Π^+_α formulae of $L_{\omega_1\omega}$ so that the recursive Σ^+_α and Π^+_α formulae will determine on any r.e. structure, respectively, Σ^0_α and Π^0_α sets. We also have the corresponding notions of Σ_α and Π_α formulae for recursive structures.

Definition 2.2.1: The Σ_0 and Π_0 formulae are the quantifier-free formulae of $L_{\omega\omega}$ (that is, finite Boolean combinations of atomic formulae).

For $\alpha > 0$, the Σ_α formulae are those of the form

$$\bigvee\!\!\!\bigvee_n \exists \mathbf{y}_n\ \psi_n(\mathbf{x},\mathbf{y}_n)$$

where $\mathbf{x}$ and each $\mathbf{y}_n$ denote finite sequences of variables and each ψ_n is a Π_β formula for some $\beta < \alpha$. Similarly, the Π_α formulae are those of the form

$$\bigwedge\!\!\!\bigwedge_n \forall \mathbf{y}_n\ \psi_n(\mathbf{x},\mathbf{y}_n)$$

where each ψ_n is a Σ_β formula for some $\beta < \alpha$.

A *basic positive formula* is a finite conjunction of formulae, each of which is either an unnegated atomic formula or a formula of the form $x_i \neq x_j$.

A *basic negative formula* is similarly a finite disjunction of formulae, each of which is either a negated atomic formula or a formula of the form $x_i = x_j$.

The Σ^+_0 and Π^+_0 formulae are just the truth values T and F.

For $\alpha > 0$, the Σ^+_α formulae are those of the form

$$\bigvee\!\!\!\bigvee_n \exists \mathbf{y}_n \; (\; \varphi_n(\mathbf{x},\mathbf{y}_n) \; \& \; \psi_n(\mathbf{x},\mathbf{y}_n) \;)$$

where $\mathbf{x}$ and each $\mathbf{y}_n$ denote finite sequences of variables, each φ_n is a basic positive formula and each ψ_n is a Π^+_β formula for some $\beta < \alpha$. Similarly, the Π^+_α formulae are those of the form

$$\bigwedge\!\!\!\bigwedge_n \forall \mathbf{y}_n \; (\; \varphi_n(\mathbf{x},\mathbf{y}_n) \vee \psi_n(\mathbf{x},\mathbf{y}_n) \;)$$

where each φ_n is a basic negative formula and each ψ_n is a Σ^+_β formula for some $\beta < \alpha$.

The difference between the two definitions is explained by the fact that on a recursive structure all atomic formulae define recursive relations, while on an r.e. structure only unnegated atomic formulae and those of the form $x_i \neq x_j$ define r.e. relations.

Definition 2.2.2: The *recursive* Σ^+_α and Π^+_α formulae are defined inductively along with their Gödel numbers as follows.

$\langle a,e,v,0\rangle$ is a Gödel number for the recursive Σ^+_a formula

$$\bigvee\!\!\!\bigvee_{n \in V} \exists \mathbf{y}_n \; (\; \varphi_n(\mathbf{x},\mathbf{y}_n) \; \& \; \psi_n(\mathbf{x},\mathbf{y}_n) \;)$$

where v is a canonical index for the finite sequence of free variables $\mathbf{x}$, and

$V = \{\ n \in W_e : \varphi_e(n) = \langle i_n, j_n \rangle$ where i_n is a Gödel number for a basic positive formula φ_n with free variables $\mathbf{x}, \mathbf{y}_n$, and $j_n = \langle b, e', v', 1 \rangle$ is a Gödel number for a recursive Π^+_b formula ψ_n with free variables $\mathbf{x}, \mathbf{y}_n$ and $b <_{\mathcal{O}} a\ \}$

Note that for $n \in W_e$ we can effectively check whether $n \in V$, so V is r.e.

Similarly $\langle a, e, v, 1 \rangle$ is a Gödel number for the recursive Π^+_a formula

$$\bigwedge\!\!\!\bigwedge_{n \in V} \forall \mathbf{y}_n\ (\ \varphi_n(\mathbf{x}, \mathbf{y}_n) \vee \psi_n(\mathbf{x}, \mathbf{y}_n)\ )$$

where v is a canonical index for the finite sequence of free variables $\mathbf{x}$, and

$V = \{\ n \in W_e : \varphi_e(n) = \langle i_n, j_n \rangle$ where i_n is a Gödel number for a basic negative formula φ_n with free variables $\mathbf{x}, \mathbf{y}_n$, and $j_n = \langle b, e', v', 0 \rangle$ is a Gödel number for a recursive Σ^+_b formula ψ_n with free variables $\mathbf{x}, \mathbf{y}_n$ and $b <_{\mathcal{O}} a\ \}$

We define a recursive Σ^+_α formula to be one which is a recursive Σ^+_a formula for some $a \in \mathcal{O}$ such that $|a| = \alpha$.

Proposition 2.2.3: Recursive Σ^+_α (respectively Π^+_α) formulae uniformly represent Σ^0_α (respectively Π^0_α) sets.

That is, if $\theta(\mathbf{x})$ is a recursive Σ^+_α (respectively Π^+_α) formula then $\{\ \mathbf{x} : \theta(\mathbf{x})\ \}$ is a Σ^0_α (respectively Π^0_α) set. Furthermore an index for such a set can be obtained uniformly from an index for the formula.

Proof: We prove the result for Σ^+_α formulae by recursive transfinite induction. The result for Π^+_α formulae follows from the facts that Π^+_α formulae are equivalent to negations of Σ^+_α formulae and Π^0_α sets are complements of Σ^0_α sets.

$\alpha = 0$: The recursive Σ^+_0 formulae are the truth values T and F.

$\alpha > 0$: Suppose $\theta(\mathbf{x}) \longleftrightarrow \bigvee\!\!\bigvee_{n\in V} \exists \mathbf{y}_n\ (\ \varphi_n(\mathbf{x},\mathbf{y}_n)\ \&\ \psi_n(\mathbf{x},\mathbf{y}_n)\ )$ is a recursive Σ^+_α formula.

We give a method of enumerating $\{\ \mathbf{x} : \theta(\mathbf{x})\ \}$ relative to $\varnothing^{(\alpha)}$ as an oracle (or $\varnothing^{(n)}$ if $\alpha = n+1 < \omega$).

Now each $\varphi_n(\mathbf{x},\mathbf{y}_n)$ is a basic positive formula, and so uniformly represents an r.e. set. Also the $\psi_n(\mathbf{x},\mathbf{y}_n)$ are $\Pi^+_{\beta_n}$ formulae for some $\beta_n < \alpha$ and so, by the induction hypothesis, uniformly represent $\Pi^0_{\beta_n}$ sets. So the oracle can determine whether

$$(\mathbf{x},\mathbf{y}_n) \in \{\ (\mathbf{x},\mathbf{y}_n) : \varphi_n(\mathbf{x},\mathbf{y}_n)\ \&\ \psi_n(\mathbf{x},\mathbf{y}_n)\ \},$$

except in the case when $\alpha = 1$, when this r.e. set can be enumerated. So enumerate W_e and as n appears check whether $n \in V$. If it is, using the oracle, start enumerating

$$\{\ (\mathbf{x},\mathbf{y}_n) : \varphi_n(\mathbf{x},\mathbf{y}_n)\ \&\ \psi_n(\mathbf{x},\mathbf{y}_n)\ \},$$

dovetailing the enumerations. For each appearance of $(\mathbf{x},\mathbf{y}_n)$ in one of these, list $\mathbf{x}$ in $\{ \mathbf{x} : \theta(\mathbf{x}) \}$.

The method as given is clearly uniform, and so allows the uniform calculation of indices. □

We have similar definitions for recursive Σ_α and Π_α formulae on recursive structures, and a parallel result to Proposition 2.2.3, which we do not state.

Definition 2.2.4: A relation R on an r.e. structure $\mathfrak{A}$ is *formally* Σ_α^{0+} if it has a definition

$$\mathfrak{A} \models \mathbf{x} \in R \longleftrightarrow \varphi(\mathbf{p},\mathbf{x})$$

where φ is a recursive Σ_α^{+} formula involving only the given (i.e r.e., by definition) relations of $\mathfrak{A}$, and $\mathbf{p}$ is a finite list of parameters from A.

The following proposition is the 'easy' direction of our main result.

Proposition 2.2.5: If a relation R is formally Σ_α^{0+} on an r.e. structure $\mathfrak{A}$, then it is intrinsically Σ_α^{0+} on $\mathfrak{A}$.

Proof: Suppose $\mathfrak{A} \models \mathbf{x} \in R \longleftrightarrow \varphi(\mathbf{p},\mathbf{x})$ where φ is a recursive Σ_α^{+} formula, and $f : \mathfrak{A} \cong \mathfrak{B}$ where $\mathfrak{B}$ is another r.e. structure. Then

$$\mathfrak{B} \models \mathbf{y} \in f(R) \longleftrightarrow \varphi'(f(\mathbf{p}),\mathbf{y})$$

where φ' is obtained from φ by replacing the r.e. relations of $\mathfrak{A}$

by the corresponding relations of $\mathfrak{B}$. So $f(R)$ has a formally Σ^{0+}_{α} definition in $\mathfrak{B}$, and by the above proposition represents a Σ^{0}_{α} set. □

§2.3 Back and Forth Relations.

Given finite sequences **a**,**b** of the same length from a structure $\mathfrak{A}$, we wish to characterize when all Π^+_α formulae true of **a** are also true of **b**. This condition is written $\Pi^+_\alpha(\mathfrak{A},\mathbf{a}) \subseteq \Pi^+_\alpha(\mathfrak{A},\mathbf{b})$. We use the following definition from [A4].

Definition 2.3.1: For sequences **a**,**b** of the same length from an r.e. structure $\mathfrak{A}$, define $\mathbf{a} \leq^+_0 \mathbf{b}$ to be invariably true. For $\alpha > 0$, define $\mathbf{a} \leq^+_\alpha \mathbf{b}$ if for every sequence **d**, every $\beta < \alpha$, and every basic positive formula φ true for **b**,**d**, there is a sequence **c**, of the same length as **d**, for which $\mathbf{a},\mathbf{c} \geq^+_\beta \mathbf{b},\mathbf{d}$ and φ is true for **a**,**c**.

Lemma 2.3.2: The following are equivalent for all $\alpha < \omega_1^{CK}$:

(i) $\mathbf{a} \leq^+_\alpha \mathbf{b}$,

(ii) $\Pi^+_\alpha(\mathfrak{A},\mathbf{a}) \subseteq \Pi^+_\alpha(\mathfrak{A},\mathbf{b})$,

(iii) $\Sigma^+_\alpha(\mathfrak{A},\mathbf{a}) \supseteq \Sigma^+_\alpha(\mathfrak{A},\mathbf{b})$.

Proof: [Ash] Clearly (ii) and (iii) are equivalent, since the negation of a Σ^+_α formula is logically equivalent to a Π^+_α formula, and vice versa.

For $\alpha = 0$, all the conditions are true from the definitions. We show that (i) is equivalent to (iii) by transfinite induction.

Suppose first that $\mathbf{a} \leq^+_\alpha \mathbf{b}$. Let $\psi(\mathbf{x})$ be any Σ^+_α formula true for **b**. Then $\psi(\mathbf{x})$ is of the form

$$\bigvee\!\!\!\!\bigvee_n \; \exists y_n \; (\; \varphi_n(\mathbf{x},\mathbf{y}_n) \; \& \; \psi_n(\mathbf{x},\mathbf{y}_n) \;)$$

where φ_n is basic positive and ψ_n is $\Pi^+_{\beta_n}$ for some $\beta_n < \alpha$. Thus, for some n and some $\mathbf{d}$, the formulae φ_n and ψ_n are true for $\mathbf{b},\mathbf{d}$. By the definition of $\leq^+_\alpha$, there therefore exists $\mathbf{c}$ for which $\mathbf{a},\mathbf{c} \geq^+_\beta \mathbf{b},\mathbf{d}$ and φ_n is true for $\mathbf{a},\mathbf{c}$. But then by the induction hypothesis, $\Pi^+_{\beta_n}(\mathfrak{A},\mathbf{a},\mathbf{c}) \supseteq \Pi^+_{\beta_n}(\mathfrak{A},\mathbf{b},\mathbf{d})$ and so also ψ_n is true for $\mathbf{a},\mathbf{c}$. Thus ψ is true for $\mathbf{a}$, as required.

Conversely, suppose that $\Sigma^+_\alpha(\mathfrak{A},\mathbf{a}) \supseteq \Sigma^+_\alpha(\mathfrak{A},\mathbf{b})$. We wish to show that $\mathbf{a} \leq^+_\alpha \mathbf{b}$, so let $\mathbf{d}$ and $\beta < \alpha$ be given and let $\varphi(\mathbf{x},\mathbf{y})$ be a basic positive formula true for $\mathbf{b},\mathbf{d}$. Consider all sequences $\mathbf{a}_i,\mathbf{c}_i$ of the same lengths as $\mathbf{b},\mathbf{d}$ for which *not* $\mathbf{a}_i,\mathbf{c}_i \geq^+_\beta \mathbf{b},\mathbf{d}$. For each i, by the induction hypothesis, there is a Π^+_β formula $\psi_i(\mathbf{x},\mathbf{y})$ which is true for $\mathbf{b},\mathbf{d}$ but false for $\mathbf{a},\mathbf{c}$. Let ψ be a Π^+_β formula equivalent to $\bigwedge\!\!\!\!\bigwedge_i \psi_i$. Then the formula $\exists \mathbf{y} \; (\; \varphi(\mathbf{x},\mathbf{y}) \; \& \; \psi(\mathbf{x},\mathbf{y}) \;)$ is a Σ^+_α formula true of $\mathbf{b}$ and so, by supposition, true of $\mathbf{a}$. So there exists $\mathbf{c}$ such that both φ and ψ are true for $\mathbf{a},\mathbf{c}$. It follows that $\mathbf{a},\mathbf{c} \geq^+_\beta \mathbf{b},\mathbf{d}$, since otherwise $\mathbf{a},\mathbf{c}$ would be $\mathbf{a}_i,\mathbf{c}_i$ for some i, and ψ is false for each $\mathbf{a}_i,\mathbf{c}_i$. We have therefore shown that $\mathbf{a} \leq^+_\alpha \mathbf{b}$. □

Ash comments that as all basic positive formulae are Π^+_2 and all basic negative formulae are Σ^+_2, the definitions of Σ^+_α (Π^+_α) formulae for $\alpha > 2$ could omit reference to basic positive (negative) formulae. The result just proved shows that the same could be done with the definition of $\leq^+_\alpha$ for $\alpha > 2$. He gives the definitions we have used here in order to avoid repetition in the proofs of the lemmas.

We have a similar situation in recursive structures, but without this complication.

Definition 2.3.3: For sequences **a**,**b** of the same length from a recursive structure $\mathfrak{A}$, define $\mathbf{a} \le_0 \mathbf{b}$ to be true if **a** and **b** satisfy the same finitary quantifier-free formulae. For $\alpha > 0$, define $\mathbf{a} \le_\alpha \mathbf{b}$ if for every sequence **d** and every $\beta < \alpha$, there is a sequence **c**, of the same length as **d**, for which $\mathbf{a},\mathbf{c} \ge_\beta \mathbf{b},\mathbf{d}$.

Lemma 2.3.4: The following are equivalent for all $\alpha < \omega_1^{CK}$:

(i) $\mathbf{a} \le_\alpha \mathbf{b}$,

(ii) $\Pi_\alpha(\mathfrak{A},\mathbf{a}) \subseteq \Pi_\alpha(\mathfrak{A},\mathbf{b})$,

(iii) $\Sigma_\alpha(\mathfrak{A},\mathbf{a}) \supseteq \Sigma_\alpha(\mathfrak{A},\mathbf{b})$.

Proof: Similar to the previous proposition. □

Corollary 2.3.5: If $\mathbf{a} \le_\alpha^+ \mathbf{b}$ (or $\mathbf{a} \le_\alpha \mathbf{b}$) then for all $\beta < \alpha$, $\mathbf{a} \le_\beta^+ \mathbf{b}$ and $\mathbf{a} \ge_\beta^+ \mathbf{b}$ (respectively $\mathbf{a} \le_\beta \mathbf{b}$ and $\mathbf{a} \ge_\beta \mathbf{b}$).

Also the relations $\le_\alpha^+$ ($\le_\alpha$) are reflexive and transitive.

Proof: The first relation is immediate from the characterizations in the Lemmas 2.3.2 and 2.3.4. The second is obtained by choosing **d** as the empty sequence in the definitions. Reflexivity and transitivity are immediate from the lemmas. □

Corollary 2.3.6: (i) $\mathbf{a} \le_\alpha \mathbf{b}$ implies $\mathbf{a} \le^+_\alpha \mathbf{b}$ for $\alpha \ge 0$.

(ii) $\mathbf{a} \le^+_{1+\alpha} \mathbf{b}$ implies $\mathbf{a} \le_\alpha \mathbf{b}$ for $\alpha \ge 1$.

Proof: These follow from Lemmas 2.3.2 and 2.3.4 and the fact that, for any $\mathfrak{A}$, $\Pi^+_\alpha(\mathfrak{A}) \subseteq \Pi_\alpha(\mathfrak{A}) \subseteq \Pi^+_{1+\alpha}(\mathfrak{A})$. The last inclusion is proved by induction on $\alpha \ge 1$. For $\alpha = 1$ it is true because basic positive formulae are Π^+_2. The induction step is trivial. □

We will be using finite partial one-one functions from an infinite recursive set B into an r.e. structure $\mathfrak{A}$ to construct a new r.e. structure $\mathfrak{B}$. Lemmas 2.3.8 and 2.3.9 are the crucial lemmas from [A4] in our context.

Definition 2.3.7: If f, g are finite partial one-one functions from B into $\mathfrak{A}$, then we define $f \angle^+_\alpha g$ if $\mathbf{d} = dom\ (f) \subseteq dom\ (g)$ and $f(\mathbf{d}) \le^+_\alpha g(\mathbf{d})$.

In the usual way, we write $f \subseteq g$ if g is an extension of f. Note that if $f \subseteq g$ then $f \angle^+_\alpha g$ for every α.

Lemma 2.3.8: Suppose that $f \angle^+_\alpha g$, $\beta < \alpha$, $\mathbf{b} \in B$ and that $\varphi(\mathbf{x})$ is a basic positive formula with $\mathfrak{A} \models \varphi(g(\mathbf{b}))$.
Then there exists a finite partial one-one $h \supseteq f$ for which $g \angle^+_\beta h$ and $\mathfrak{A} \models \varphi(h(\mathbf{b}))$.

Proof: We may suppose that $dom\ (g) = \mathbf{b} = \mathbf{b}_1,\mathbf{b}_2$ where $\mathbf{b}_1 = dom\ (f)$. Let $f(\mathbf{b}_1) = \mathbf{a}_1$, $g(\mathbf{b}_1) = \mathbf{c}_1$, $g(\mathbf{b}_2) = \mathbf{c}_2$. Then since $f \angle^+_\alpha g$ we have

$\mathbf{a}_1 \leq^+_\alpha \mathbf{c}_1$. Let φ' be the conjunction of φ with all the inequalities which hold for $\mathbf{c}_1,\mathbf{c}_2$. Then φ' is also a basic positive formula and, since $\mathfrak{A} \models \varphi'(\mathbf{c}_1,\mathbf{c}_2)$, by definition of $\leq^+_\alpha$, there exists $\mathbf{a}_2 \in \mathfrak{A}$ for which $\mathbf{a}_1,\mathbf{a}_2 \geq^+_\beta \mathbf{c}_1,\mathbf{c}_2$ and $\mathfrak{A} \models \varphi'(\mathbf{a}_1,\mathbf{a}_2)$. We may therefore define h by $h(\mathbf{b}_1,\mathbf{b}_2) = \mathbf{a}_1,\mathbf{a}_2$. □

Lemma 2.3.9: Suppose that for each $i = 1,2,...,k$, f_i is a finite one-one partial function from B to $\mathfrak{A}$. Suppose also that $\xi_1 > \xi_2 > ... > \xi_k \geq 0$, that $f_1 \angle^+_{\xi_1} f_2 \angle^+_{\xi_2} \dots \angle^+_{\xi_{k-1}} f_k$, and that $\mathfrak{A} \models \varphi(f_k(\mathbf{b}))$ where $\mathbf{b} \in B$ and $\varphi(\mathbf{x})$ is a basic positive formula.

Then there is a finite partial one-one function g from B to $\mathfrak{A}$ for which $g \supseteq f_1$, $\mathfrak{A} \models \varphi(g(\mathbf{b}))$ and $f_i \angle^+_{\xi_i} g$ for each $i = 1,2,...,k$.

Proof: By induction on k. For $k = 1$, take $g = f_1$. For $k = 2$, the result is given by Lemma 2.3.8.

For $k > 2$, by Lemma 2.3.8 there exists $h \supseteq f_{k-1}$ for which $f_k \angle^+_{\xi_k} h$ and $\mathfrak{A} \models \varphi(h(\mathbf{b}))$.

Then $f_{k-2} \angle^+_{\xi_{k-1}} f_{k-1} \subseteq h$, so we have

$$f_1 \angle^+_{\xi_1} f_2 \angle^+_{\xi_2} \dots \angle^+_{\xi_{k-3}} f_{k-2} \angle^+_{\xi_{k-2}} h.$$

Thus, by the induction hypothesis, there exists $g \supseteq f_1$ such that $f_i \angle^+_{\xi_i} g$ for $i = 1,2,...,k-2$, $h \angle^+_{\xi_{k-1}} g$ and $\mathfrak{A} \models \varphi(g(\mathbf{b}))$.

But then $f_{k-1} \subseteq h \angle^+_{\xi_{k-1}} g$, which gives $f_{k-1} \angle^+_{\xi_{k-1}} g$, and $f_k \angle^+_{\xi_k} h \angle^+_{\xi_{k-1}} g$, which gives $f_k \angle^+_{\xi_k} g$. □

§2.4 More about Recursive Infinitary Formulae.

The definition given above of recursive Σ^+_α and Π^+_α formulae was a little more restrictive than it needed to be.

Lemma 2.4.1: Suppose S is a Σ^0_α set, and for each $s \in S$, φ_s is a basic positive (respectively negative) formula and ψ_s is a recursive Π^+_β (respectively Σ^+_β) formula for some $\beta < \alpha$, then

$$\bigvee\!\!\!\!\bigvee_{s\in S} \exists y_s \, (\varphi_s(x,y_s) \,\&\, \psi_s(x,y_s))$$

$$(\text{ respectively } \bigwedge\!\!\!\!\bigwedge_{s\in S} \forall y_s \, (\varphi_s(x,y_s) \vee \psi_s(x,y_s)) \quad)$$

is equivalent to a recursive Σ^+_α (respectively Π^+_α) formula. A Gödel number for such a formula can be found uniformly in an index for S.

Proof: We first prove by recursive transfinite induction on α that for each Σ^0_α set S there is a recursive function which assigns to each $s \in \mathbb{N}$ an index for a Σ^+_α sentence χ_s for which

$$s \in S \;\Rightarrow\; \chi_s \text{ is logically valid.}$$

$$s \notin S \;\Rightarrow\; \neg\chi_s \text{ is logically valid.}$$

$\alpha = 1$: Then $S = W_e$ for some e. Take as χ_s

$$\bigvee\!\!\!\!\bigvee_{y,\,t\in\mathbb{N}} \exists x \; \theta_{syet}$$

where θ_{syet} is $\begin{cases} x=x & \text{if } \varphi^t_e(s) = y \\ x\neq x & \text{otherwise.} \end{cases}$

$\alpha > 1$: $S = W_{e,a}$ is r.e. relative to H where

$$H = \{ \langle m,e',b\rangle : b <_{\mathcal{O}} a \ \&\ m \in W_{e',b} \}$$

So $s \in S$ can be expressed as an infinite, r.e. disjunction of finite conjunctions of statements of the form $n \in H$ and $n \notin H$. These disjuncts represent all sets of oracle responses which yield a terminating computation. Each statement $n \in H$ may be effectively re-expressed as $m \in W_{e',b}$ for some $b <_{\mathcal{O}} a$, so by the induction hypothesis as a recursive Σ^+_β sentence for some $\beta < \alpha$.

Each finite Boolean combination of such sentences may be effectively re-expressed as a Σ^+_α sentence, as may an r.e. disjunction of Σ^+_α sentences.

With this we see that $\bigvee\!\!\bigvee_{s\in S} \exists y_s\ (\ \varphi_s(x,y_s)\ \&\ \psi_s(x,y_s)\ )$ may be re-expressed as $\bigvee\!\!\bigvee_{s\in\mathbb{N}} [\ \chi_s\ \&\ \exists y_s\ (\ \varphi_s(x,y_s)\ \&\ \psi_s(x,y_s)\ )\ ]$, which is equivalent to a recursive Σ^+_α formula.

Similarly $\bigwedge\!\!\bigwedge_{s\in S} \forall y_s\ (\ \varphi_s(x,y_s) \vee \psi_s(x,y_s)\ )$ may be re-expressed as $\bigwedge\!\!\bigwedge_{s\in\mathbb{N}} [\ \chi_s \longrightarrow \forall y_s\ (\ \varphi_s(x,y_s) \vee \psi_s(x,y_s)\ )\ ]$. $\square$

The corresponding result is also true for recursive Σ_α and Π_α formulae.

In well behaved structures the existence of a Π^+_α formula with certain properties will imply the existence of a recursive Π^+_α formula with those properties.

Definition 2.4.2: A *positive open formula* is a finite disjunction of basic positive formulae. A *positive existential formula* is a (finitary) formula of the form $\exists \mathbf{y}\ \varphi(\mathbf{x},\mathbf{y})$ where φ is a positive open formula.

The *positive open diagram* (respectively *positive existential diagram*) of a structure $\mathfrak{A} = (A,\langle R_i\rangle_{i\in I})$ is the set of positive open (respectively existential) sentences of $L(A)$ which are true in $(\mathfrak{A},\langle a\rangle_{a\in A})$.

Lemma 2.4.3: Suppose that the relations $\leq^{+}_{\beta}$ $(1 \leq \beta < \alpha)$ are uniformly Σ^{0}_{β} and that the positive existential diagram of $\mathfrak{A}$ is recursive. Then for each $\beta \leq \alpha$ and each sequence $\mathbf{a}$, we can effectively find a recursive Π^{+}_{β} formula $\psi^{\mathbf{a}}_{\beta}(\mathbf{x})$ such that for all sequences $\mathbf{b}$ we have

$$\mathbf{a} \leq_{\beta} \mathbf{b} \text{ iff } \mathfrak{A} \models \psi^{\mathbf{a}}_{\beta}(\mathbf{b}).$$

Proof: [Ash] For $\beta = 0$, $\psi^{\mathbf{a}}_{0}$ may be taken to be the truth value T.

For $\beta > 0$, we proceed by recursive transfinite induction on $\beta \leq \alpha$ to define an index for a suitable $\psi^{\mathbf{a}}_{\beta}$. We have from the definition that $\mathbf{a} \leq^{+}_{\beta} \mathbf{b}$ iff for every $\gamma < \beta$, every sequence $\mathbf{d}$ and every basic positive formula $\varphi(\mathbf{x},\mathbf{y})$ for which $\mathfrak{A} \models \varphi(\mathbf{b},\mathbf{d})$, there exists $\mathbf{c}$ for which $\mathbf{a},\mathbf{c} \geq^{+}_{\gamma} \mathbf{b},\mathbf{d}$ and $\mathfrak{A} \models \varphi(\mathbf{a},\mathbf{c})$.

For fixed $\beta > 0$ and $\mathbf{a}$, and for each $\gamma < \beta$, let S_{γ} denote the set of triples $\sigma = (\mathbf{b},\mathbf{d},\varphi)$ for which $\mathbf{b}$ has the same length as $\mathbf{a}$, $\varphi(\mathbf{x},\mathbf{y})$ is a basic positive formula, $\mathfrak{A} \models \varphi(\mathbf{b},\mathbf{d})$ and yet there is no $\mathbf{c}$ for which both $\mathbf{a},\mathbf{c} \geq^{+}_{\gamma} \mathbf{b},\mathbf{d}$ and $\mathfrak{A} \models \varphi(\mathbf{a},\mathbf{c})$. Then

$$\bigwedge\!\!\!\bigwedge_{\gamma<\beta} \ \bigwedge\!\!\!\bigwedge_{\sigma\in S_\gamma} \forall \mathbf{y} \ (\ \neg\varphi(\mathbf{x},\mathbf{y}) \vee \neg\psi_\gamma^{\mathbf{b},\mathbf{d}}(\mathbf{x},\mathbf{y}) \ )$$

has the required property. It remains to see that this is equivalent to a recursive Π^+_β formula.

By our assumptions, the conditions $\mathfrak{A} \models \varphi(\mathbf{a},\mathbf{c})$ and $\mathfrak{A} \models \varphi(\mathbf{b},\mathbf{d})$ are, for basic positive φ, recursive, and the condition *not* $\mathbf{a},\mathbf{c} \geq^+_\gamma \mathbf{b},\mathbf{d}$ is uniformly Π^0_γ. Thus for $\gamma \geq 1$, the set S_γ is Π^0_γ uniformly in γ, and so, by Lemma 2.4.1, the inner conjunct is equivalent to a recursive $\Pi^+_{\gamma+1}$ formula.

For $\gamma = 0$, the assumption that the positive existential diagram of $\mathfrak{A}$ is recursive ensures that S_0 is recursive (the condition $\mathbf{a},\mathbf{c} \geq^+_0 \mathbf{b},\mathbf{d}$ is invariably true), so the inner conjunct is equivalent to a Π^+_1 formula.

Consequently the whole formula may be re-expressed as a Π^+_β formula as required. □

As noted in [A4], the assumptions in the above Lemma imply that $\mathfrak{A}$ is in fact a recursive structure. We may still, however, construct isomorphic r.e. copies which are not necessarily recursive. Ash comments in [A4]:

"The need for such assumptions, here as elsewhere, seems to reflect the fact that for the recursive properties of r.e. or recursive copies of a structure to correlate neatly with the recursive syntactical properties of the structure, there must be at least one copy of the structure in which many additional features are recursive."

We can now give a simple sufficient condition for a relation to be formally Σ^{0+}_{α}.

Definition 2.4.4: Given an r.e. structure $\mathfrak{A}$ and a new relation R on $\mathfrak{A}$, we define for each finite sequence $\mathbf{p}$ from A and each $2 \le \alpha < \omega_1^{CK}$, the subset $Rcl^+_{\alpha}(\mathbf{p})$ of R:

If $\alpha = \beta+1$, then $r \in Rcl^+_{\alpha}(\mathbf{p})$ if for some $\mathbf{b}$, whenever $\mathbf{p},r,\mathbf{b} \le^+_{\beta} \mathbf{p},r',\mathbf{b}'$ then $r' \in R$.

If α is a limit ordinal, then $Rcl^+_{\alpha}(\mathbf{p}) = \bigcup_{\beta<\alpha} Rcl^+_{\beta}(\mathbf{p})$.

Proposition 2.4.5: Let $\mathfrak{A}$ be an r.e. structure, R be a new relation on $\mathfrak{A}$, and $2 \le \alpha < \omega_1^{CK}$. Suppose the relations $\le^+_{\beta}$ are uniformly Σ^0_{β} for $\beta < \alpha$, and the positive existential diagram of $\mathfrak{A}$ and R are recursive. If, for some $\mathbf{p}$, $Rcl^+_{\alpha}(\mathbf{p}) = R$, then R is formally Σ^{0+}_{α}.

Proof: We first deal with the successor ordinal case $\alpha = \beta+1$. We have that $R = Rcl^+_{\alpha}(\mathbf{p})$, so for each $r \in R$ there is $\mathbf{b}$ such that whenever $\mathbf{p},r,\mathbf{b} \le^+_{\beta} \mathbf{p},r',\mathbf{b}'$ then $r' \in R$. This relation between $\mathbf{p},r,\mathbf{b}$ is Π^0_{β}, so a suitable $\mathbf{b}_r$, depending on r, can be found by a Δ^0_{α} process.

Let $\psi^{\mathbf{p},r,\mathbf{b}_r}_{\beta}(\mathbf{x},y,\mathbf{z}_r)$ be the formula guaranteed by Lemma 2.4.3. Then

$$\mathfrak{A} \models y \in R \longleftrightarrow \bigvee\!\!\!\bigvee_{r\in R} \exists \mathbf{z}_r\, \psi^{\mathbf{p},r,\mathbf{b}_r}_{\beta}(\mathbf{x},y,\mathbf{z}_r).$$

The disjunction in this formula is Σ^0_α (we have to find an appropriate $\mathbf{b}_r$ for each $r \in R$), so by Lemma 2.4.1 it is equivalent to a recursive Σ^+_α formula.

Now suppose α is a limit ordinal. Then for each $r \in R$ there is $\beta < \alpha$ and $\mathbf{b}$ such that whenever $\mathbf{p},r,\mathbf{b} \leq^+_\beta \mathbf{p},r',\mathbf{b}'$ then $r' \in R$. This relation between $\mathbf{p},r,\mathbf{b}$ is Π^0_β, so a suitable $\beta_r < \alpha$ and $\mathbf{b}_r$, depending on r, can be found by a Δ^0_α process. Again

$$\mathfrak{A} \models y \in R \longleftrightarrow \bigvee\!\!\!\bigvee_{r \in R} \exists \mathbf{z}_r \, \psi^{\mathbf{p},r,\mathbf{b}_r}_{\beta_r}(\mathbf{x},y,\mathbf{z}_r)$$

and by Lemma 2.4.1 this is equivalent to a recursive Σ^+_α formula. □

§2.5 Labelling Systems.

We give here the definitions from [A4] which are needed for the proof of our main result.

Definition 2.5.1: An *r.e. scheme* is a structure $\mathfrak{S} = (U,L,P,E)$ in which U, L are r.e. sets, P is an r.e. set of finite sequences of the form $(u_0,l_0,u_1,l_1,\ldots)$ where each $u_i \in U$ and each $l_i \in L$, and $E \subseteq L \times \mathbb{N}$ is r.e. Also we require that P is closed under the formation of initial segments.

An *instruction* for $\mathfrak{S}$ is a function p which assigns, to each member of P of the form $(u_0,l_0,\ldots u_n,l_n)$, an element $u_{n+1} \in U$ for which $(u_0,l_0,\ldots u_n,l_n,u_{n+1}) \in P$.

An *input* for $\mathfrak{S}$ is a triple (u,l,p) in which p is an instruction for $\mathfrak{S}$ and $(u,l) \in P$. We say that (u,l,p) is Δ^0_α if p is.

A *run* in $\mathfrak{S}$ of the input (u,l,p) is an infinite sequence

$$s = (u_0,l_0,u_1,l_1,\ldots)$$

of which every finite initial segment is in P, such that $u_0 = u$, $l_0 = l$ and such that, for each n, $u_{n+1} = p(u_0,l_0,\ldots u_n,l_n)$.

For $l \in L$ we write $E(l) = \{\ m : (l,m) \in E\ \}$ and if $s = (u_0,l_0,u_1,l_1,\ldots)$ we write $E(s) = \bigcup_n E(l_n)$. We say that the run s is *conservative* if $E(s)$ is r.e.

This scheme corresponds to an infinite two player game in which, alternately, player I plays elements of U and player II plays elements of L, so that at each stage the corresponding finite sequence remains in P.

An input corresponds to a starting position and a strategy for player I. Player II is considered to win against this strategy if the infinite sequence $l_0,l_1,...$ of his plays makes $\bigcup_n E(l_n)$ r.e.

Ash's name "labelling system" arises from his original visualization of the situation. One thinks of a tree with nodes from U, to be labelled in a run of the game by labels from L. Thus, each turn of player I consists in choosing the next node; each reply of player II consists in choosing a label for that node. Player II wins a run of the game s if it is conservative, that is if his choice of labels has forced $E(s)$ to be r.e.

To apply this, we will design r.e. schemes for which the existence of a conservative run for a suitable input establishes the desired results. So we wish to ensure that player II can win when player I follows a certain strategy. This strategy consists in supplying information from a Δ^0_α oracle about the Σ^0_α sets. To this end, we will want to prove that player II has a winning strategy against any Δ^0_α strategy of player I.

Definition 2.5.2: An *α-system* (for $1 \le \alpha \le \omega_1^{CK}$) is a structure $\mathcal{S} = (U,L,P,E,\langle\le_\gamma\rangle_{\gamma<\alpha})$ for which (U,L,P,E) is an r.e. scheme, each $\le_\gamma$ is, uniformly in $\gamma < \alpha$, an r.e. binary relation on L, and the following conditions are satisfied.

(1) Each $\le_\gamma$ is reflexive and transitive.

(2) If $\gamma < \beta < \alpha$ and $l \subseteq_\beta m$, then $l \subseteq_\gamma m$.

(3) If $l \subseteq_0 m$ then $E(l) \subseteq E(m)$.

(4) If $(u_0,l_0,...u_n,l_n,u_{n+1}) \in P$, if $0 \leq \gamma_k <...< \gamma_2 < \gamma_1 < \alpha$ and if $l_n = m_1 \subseteq_{\gamma_1} m_2 \subseteq_{\gamma_2} ... \subseteq_{\gamma_{k-1}} m_k$ then there exists l_{n+1} for which $(u_0,l_0,...u_n,l_n,u_{n+1},l_{n+1}) \in P$ and $m_i \subseteq_{\gamma_i} l_{n+1}$ for each $i = 1,2,...,k$.

The sets $E(l)$ will represent the finite portion of the positive open diagram of the structure we will be building that is enumerated by the time the game reaches the node with label l. So $E(s) = \bigcup_n E(l_n)$ represents the complete enumeration of the positive open diagram. A conservative run thus amounts to the construction of an r.e. structure.

The condition (4) is the crucial condition which provides the winning strategy for player II. The construction amounts to an α-level injury priority argument. The Δ^0_α oracle represents the information provided by an iterated limit of recursive functions. Since the enumerated positive open diagram is to be r.e., once a formula is enumerated it cannot be changed. Condition (4) allows the construction the necessary flexibility to adapt to the changing information 'provided by the oracle'.

Theorem 2.5.3: [Ash] Every Δ^0_α input for an α-system has a conservative run.

Proof: See [A1], [A2] and [A4]. ■

Ash's α-systems are related to constructions by Julia Knight in [Kn1] and [Kn2].

§2.6 Main Theorem.

We can now use Theorem 2.5.3 to prove the following.

Theorem 2.6.1: Let $2 \leq \alpha < \omega_1^{CK}$, $\mathfrak{A}$ be an r.e. structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:

(1) The positive existential diagram of $\mathfrak{A}$ is recursive.

(2) R is a recursive set.

(3) The relations $\leq_\beta^+$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which finds, given $\mathbf{a}$, some

$$r \in R - Rcl_\alpha^+(\mathbf{a}).$$

Then R is not intrinsically Σ_α^{0+}.

Proof: We consider the case where R is a unary relation. The general case may be proved from this by considering a suitable product structure, obtained effectively from the given structure, on which the relation corresponding to R is unary.

We want to construct an r.e. structure isomorphic to $\mathfrak{A}$ in which the relation corresponding to R is not a Σ_α^0 set.

So take as the domain of our new structure $\mathfrak{B}$, a fixed infinite recursive set B (say $B = \mathbb{N}$). We will construct an isomorphism $f : \mathfrak{B} \cong \mathfrak{A}$ as the union of a chain of finite partial functions defined on subsets of B.

Let $\{a_i\}$, $\{b_i\}$ be fixed r.e. enumerations of $dom\ \mathfrak{A}$ and B.

We will decide sentences in the (atomic) diagram of $\mathfrak{B}$ in the course of the construction. To ensure that $\mathfrak{B}$ is an r.e. structure, these sentences will not be changed once decided. We suppose we have a fixed enumeration of the diagram of $\mathfrak{A}$.

To ensure that $f^{-1}(R)$ is not a Σ^0_α set, we want to satisfy the requirements

$$Req_e : \quad f^{-1}(R) \neq W_{e,a}$$

where $|a|_{\mathcal{O}} = \alpha$, so $W_{e,a}$ is the e^{th} Σ^0_α set.

The idea of the construction is to choose for each e a suitable witness $r \in R - Rcl^+_\alpha(\mathbf{a})$ for Req_e, where $\mathbf{a}$ represents the image of the part of the construction with higher priority.

We choose $m \in B$ and plan to map m to r as long as m is not enumerated in the e^{th} Σ^0_α set. If m is enumerated into this set then we will be able to switch to mapping m to some element of $\mathfrak{A}$ not in R without having to change the previously enumerated portion of the diagram of $\mathfrak{B}$. (This will be possible because of our choice of r not in $Rcl^+_\alpha(\mathbf{a})$.)

We construct an α-system to carry out the construction relative to a Δ^0_α oracle.

Let $U = \{0,1\}$. The intention is that $u_{i+1} = 1$ (respectively 0) should mean, on the true path of the construction, that m is (respectively, is not) enumerated in the e^{th} Σ^0_α set by stage s, where $i = \langle m,e,s \rangle$.

Let L be the set of finite sequences of the form

$$(f_0, x_0, \dots, f_h, x_h, D)$$

where for each i, f_i is a finite one-one partial function from B to $\mathfrak{A}$, $f_i \subseteq f_{i+1}$, $x_i \in \{*\} \cup R - Rcl^+_\alpha(ran(f_i))$, $x_i, a_i \in \{*\} \cup ran(f_{i+1})$ and $b_i \in dom(f_{i+1})$. Also $x_h \neq *$, and D is (the set of Gödel numbers of) a finite set of atomic sentences $\varphi(\mathbf{b})$ of $L(B)$ for which $\mathbf{b} \in dom(f_h)$ and $\mathfrak{A} \models \varphi(f_h(\mathbf{b}))$.

Let P be the set of finite sequences $(u_0, l_0, u_1, l_1, \dots)$ where $l_i = (f_{0i}, x_{0i}, \dots, f_{h_i i}, x_{h_i i}, D_i)$ such that, for each i,

(i) $D_i \subseteq D_{i+1}$

(ii) D_i contains each atomic sentence $\varphi(\mathbf{b})$ of $L(B)$ for which $\varphi(f_{h_i i}(\mathbf{b}))$ has appeared in the first i stages of the enumeration of the diagram of $\mathfrak{A}$.

(iii) If $u_{i+1} = 1$, $e < h_i$ and $f_{(e+1)i}(m) = x_{ei}$ where $i = \langle m,e,s \rangle$, then

$$l_{i+1} = (f_{0(i+1)}, x_{0(i+1)}, \dots, f_{e(i+1)}, *, f_{(e+1)(i+1)}, x_{(e+1)(i+1)}, D_{i+1})$$

where $f_{j(i+1)} = f_{ji}$ and $x_{j(i+1)} = x_{ji}$ for $j \leq e$, and $f_{(e+1)(i+1)}(m) \notin R$.

(iv) Otherwise (i.e. if $u_{i+1} = 0$ or $f_{(e+1)i}(m) \neq x_{ei}$)

$$l_{i+1} = (f_{0(i+1)}, x_{0(i+1)}, \ldots, f_{h_i(i+1)}, x_{h_i(i+1)},$$
$$f_{(h_i+1)(i+1)}, x_{(h_i+1)(i+1)}, D_{i+1})$$

where $f_{j(i+1)} = f_{ji}$ and $x_{j(i+1)} = x_{ji}$ for $j \leq h_i$.

If $l = (f_0, x_0, \ldots, f_h, x_h, D)$ we define $E(l) = D$.
If $l' = (f'_0, x'_0, \ldots, f'_h, x'_h, D')$ we define $l \leq_\xi l'$ iff $D \subseteq D'$ and $f_h \angle^+_\xi f'_{h'}$.

Note that in the situation in (iv) l_{i+1} extends l_i; in (iii) l_{i+1} discards some of the functions in l_i but retains those corresponding to higher priority requirements than the Req_e being attended to. The * here is intended to show that Req_e has been definitely satisfied (unless subsequently injured by a higher priority requirement), as m has been enumerated into $W_{e,a}$ but we have arranged that its image is not in R.

Lemma 2.6.2: This forms an α-system.

Proof: L is r.e. because of conditions (1) and (4). P is r.e. because we can effectively check whether two finite partial functions are the same and we can effectively check whether $x \notin R$ by condition (2).

(1) and (2) of Definition 2.5.2 are immediate from the definitions and Corollary 2.3.5. (3) is immediate from the definitions, so it remains to check (4) of Definition 2.5.2.

To establish this, suppose that $(u_0, l_0, \ldots, u_n, l_n, u_{n+1}) \in P$, $\alpha > \xi_1 > \xi_2 > \ldots > \xi_k \geq 0$ and $l_n = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \ldots \subseteq_{\xi_{k-1}} m_k$ where $l_n = (f_{0n}, x_{0n}, \ldots, f_{h_n n}, x_{h_n n}, D_n)$ and $m_j = (\ldots, f'_j, x'_j, D'_j)$.

Then we have by definition

$$f_{h_n n} = f'_1 \angle^+_{\xi_1} f'_2 \angle^+_{\xi_2} \ldots \angle^+_{\xi_k} f'_k \quad \text{and} \quad D'_1 \subseteq D'_2 \subseteq \ldots \subseteq D'_k.$$

Let $\varphi(\mathbf{b})$ be the conjunction of the sentences of D'_k. Then, since $m_k \in L$, $\mathfrak{A} \models \varphi(f'_k(\mathbf{b}))$. So, by Lemma 2.3.9, there exists $g \supseteq f'_1$ such that $\mathfrak{A} \models \varphi(g(\mathbf{b}))$ and $f'_j \angle^+_{\xi_j} g$ for each $j = 1,2,\ldots,k$.

In the case where $u_{n+1} = 0$ or $f_{(e+1)n}(m) \neq x_{en}$, we may take the required l_{n+1} to be $(f_{0n}, x_{0n}, \ldots, f_{h_n n}, x_{h_n n}, g', x', D')$ where $g' \supseteq g$ and $D' \supseteq D'_k$ are chosen to satisfy the definitions of L and P, and $x' \in R - Rcl^+_\alpha(\, ran(g') \,)$.

In the other case, where $u_{i+1} = 1$, $i = \langle m,e,s \rangle$, $e < h_i$ and $f_{(e+1)i}(m) = x_{ei}$, we have $g \supseteq f_{h_n n} \supseteq f_{en}$.

Let $ran(g) = \mathbf{a}, x_{en}, \mathbf{b}$ where $ran(f_{en}) = \mathbf{a}$. Now $x_{en} \in R - Rcl^+_\alpha(\mathbf{a})$, so there are x' and $\mathbf{b}'$ with $\mathbf{a}, x_{en}, \mathbf{b} \leq^+_{\xi_1} \mathbf{a}, x', \mathbf{b}'$, $\mathfrak{A} \models \varphi(\mathbf{a}, x', \mathbf{b}')$ and $x' \notin R$.

Define $g' : dom(g) \longrightarrow \mathfrak{A}$ by $g' \circ g^{-1}(\mathbf{a}, x_{en}, \mathbf{b}) = \mathbf{a}, x', \mathbf{b}'$.

Now choose $g'' \supseteq g'$ and $D' \supseteq D'_k$ to satisfy the definitions of L and P, and $x'' \in R - Rcl^+_\alpha(ran(g''))$.

Then we may take the required l_{n+1} to be

$$(f_{0n}, x_{0n}, \ldots, f_{en}, *, g'', x'', D').$$

Thus we have defined an α-system. $\square$

Now there is a Δ^0_α function p' with $p'(i) = 1$ (respectively 0) iff m is (respectively, is not) enumerated in the e^{th} Σ^0_α set by stage s, where $i = \langle m,e,s\rangle$. So consider the input $(0,l,p)$ where $l = (\emptyset, x, \emptyset)$, $x \in R - Rcl^+_\alpha(\emptyset)$ and $p(u_0,l_0,\ldots,u_i,l_i) = p'(i)$.
By Theorem 2.5.3, this input has a conservative run $(u_0,l_0,\ldots)$.
In this conservative run, let $l_i = (f_{0i}, x_{0i}, \ldots, f_{h_i i}, x_{h_i i}, D)$.

Lemma 2.6.3: For each n, $lim_i\ f_{ni} = f_n$ and $lim_i\ x_{ni} = x_n$ exist, and the f_n form a chain.

Proof: By induction on n. From the definition of P we see that $f_{0i} = f_{0(i+1)}$ for each i. Also $f_{1i} = f_{1(i+1)}$ and $x_{0i} = x_{0(i+1)}$ unless $u_{i+1} = 1$, $i = \langle m,e,s\rangle$, $e = 0$ and $f_{1i}(m) = x_{0i}$ for some i. In this case $x_{0(i+1)} = *$, and neither it nor $f_{1(i+1)}$ ever changes again.

The inductive step is the same. Assume that we have reached a stage j by which all the $x_{0j}, f_{1j}, \ldots, x_{(n-1)j}, f_{nj}$ have reached their limiting values and do not hereafter change. Then $f_{(n+1)i} = f_{(n+1)(i+1)}$ and $x_{ni} = x_{n(i+1)}$ for all $i \geq j$ unless $u_{i+1} = 1$, $i = \langle m,e,s\rangle$, $e = n$ and $f_{(n+1)i}(m) = x_{ni}$ for some $i \geq j$. In this case $x_{n(i+1)} = *$, and neither it nor $f_{(n+1)(i+1)}$ ever changes again.

The f_n form a chain because the f_{ni} form a chain for each i. $\square$

Lemma 2.6.4: Each requirement Req_e is satisfied and $f = \bigcup_n f_n$ is an isomorphism from B to $\mathfrak{A}$.

Proof: If $x_e = *$, then for some m we have $f(m) = f_{e+1}(m) \notin R$, but $m \in W_{e,a}$. On the other hand, if $x_e \neq *$, then for some m we have $f(m) = f_{e+1}(m) = x_e \in R$ but $m \notin W_{e,a}$.

Since f is the union of a chain of one-one functions it is also one-one, and the construction ensures that each member of B and *dom* $\mathfrak{A}$ is used in the definition of f.

The finite sets of atomic sentences of $L(B)$ form a chain whose union is the diagram of the desired r.e. structure $\mathfrak{B}$. The construction guarantees that each f_n is a finite partial isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, so f is an isomorphism. □

Thus we have constructed an r.e. structure $\mathfrak{B} \cong \mathfrak{A}$ and an isomorphism f for which $f^{-1}(R)$ is not a Σ^0_α set. So R is not intrinsically Σ^{0+}_α on $\mathfrak{A}$. ∎

We can combine this with Propositions 2.2.5 and 2.4.5 to yield:

Theorem 2.6.5: Let $2 \leq \alpha < \omega_1^{CK}$, $\mathfrak{A}$ be an r.e. structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:

(1) The positive existential diagram of $\mathfrak{A}$ is recursive.

(2) R is a recursive set.

(3) The relations $\leq_\beta$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which determines for each x and $\mathbf{a}$ whether $x \in Rcl^+_\alpha(\mathbf{a})$.

Then R is intrinsically Σ^{0+}_α iff it is formally Σ^{0+}_α.

Proof: By Proposition 2.2.5, R is intrinsically Σ^{0+}_α if it is formally Σ^{0+}_α, so suppose R is not formally Σ^{0+}_α. Then by Proposition 2.4.5, for every $\mathbf{a} \in \mathfrak{A}$, there is $r \in R - Rcl^+_\alpha(\mathbf{a})$. We can use the procedure in condition (4) and the fact that R is recursive to effectively find such an r, as required by condition (4) of Theorem 2.6.1. Then we may apply the theorem to conclude that R is not intrinsically Σ^{0+}_α on $\mathfrak{A}$. ∎

§2.7 Extensions.

We may prove the corresponding result to Theorem 2.6.1 for recursive structures from that theorem as follows.

Theorem 2.7.1: Let $2 \leq \alpha < \omega_1^{CK}$, $\mathfrak{A}$ be a recursive structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:

(1) The existential diagram of $\mathfrak{A}$ is recursive.

(2) R is a recursive set.

(3) The relations $\leq_\beta$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which finds, given $\mathbf{a}$, some

$$r \in R - Rcl_\alpha(\mathbf{a}).$$

Then R is not intrinsically Σ^0_α.

Proof: The recursive structure $\mathfrak{A} = (A,\langle R_i\rangle_{i\in I})$ corresponds to an r.e. structure $\mathfrak{A}' = (A,\langle R_i\rangle_{i\in I},\langle \bar{R}_i\rangle_{i\in I})$. Each Σ_β formula of $L(\mathfrak{A})$ corresponds to an equivalent Σ^+_β formula of $L(\mathfrak{A}')$ by replacing appropriate occurrences of the symbols $\neg R_i$ and R_i by the new symbols $\bar{R}_i$ and $\neg\bar{R}_i$ respectively. This transformation is reversible and effective, so that recursive Σ_β formulae correspond to recursive Σ^+_β formulae. Also the finitary quantifier-free and existential formulae of $L(\mathfrak{A})$ correspond effectively to equivalent finitary positive and positive existential formulae of $L(\mathfrak{A}')$ respectively.

So the conditions (1)-(4) above imply the corresponding conditions in Theorem 2.6.1. Thus there is an r.e. structure $\mathfrak{B}'$ isomorphic to $\mathfrak{A}'$ in which the corresponding relation to R is not a Σ^0_α set. But then the reduct $\mathfrak{B}$ of $\mathfrak{B}'$ is an appropriate recursive structure. ■

Using this we also have the corresponding result to Theorem 2.6.5 for recursive structures.

A generalization of Theorem 2.6.1 that may be proved in essentially the same way is the following.

Theorem 2.7.2: Let $2 \leq \alpha < \omega_1^{CK}$, $\mathfrak{A}$ be an r.e. structure and $\langle R_j \rangle_{j \in J}$ a finite or countable set of new relations on $\mathfrak{A}$ satisfying the following conditions:

(1) The positive existential diagram of $\mathfrak{A}$ is recursive.

(2) Each R_j is a recursive set, uniformly in $j \in J$.

(3) The relations $\leq^+_\beta$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which finds, given $j \in J$ and

$$\mathbf{a}, \text{ some } r \in R_j - R_j cl_\alpha(\mathbf{a}).$$

Then there is an r.e. structure $\mathfrak{B}$, isomorphic to $\mathfrak{A}$, on which the image of R_j is not Σ^0_α, for each $j \in J$.

Proof: In the construction of Theorem 2.6.1 we satisfy countably many requirements to ensure that $f^{-1}(R)$ is not a Σ^0_α set. Clearly we can interweave the requirements for each R_j and satisfy them all, provided that we are assured that the construction will not break down. Condition (4) gives us that assurance. ■

As a particular case of this we have

Proposition 2.7.3: Let $2 \leq \alpha < \omega_1^{CK}$, $\mathfrak{A}$ be an r.e. structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:

(1) The positive existential diagram of $\mathfrak{A}$ is recursive.

(2) R is a recursive set.

(3) The relations $\leq_\beta^+$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which finds, given **a**, some

$$r \in R - Rcl_\alpha(\mathbf{a}) \text{ and some } r' \in \bar{R} - \bar{R}cl_\alpha(\mathbf{a}).$$

Then there is an r.e. structure $\mathfrak{B}$, isomorphic to $\mathfrak{A}$, on which the image of R is not Σ_α^0 or Π_α^0. □

Of course, the corresponding results for recursive structures are also true.

§2.8 A Related Notion.

A natural class of sets has not been given a name in the hyperarithmetical hierarchy. For limit ordinals α, let

$$\hat{\Sigma}^0_\alpha = \bigcup_{\beta<\alpha} \Sigma^0_\beta.$$

We can give similar results to Theorems 2.6.1 and 2.6.5 involving this notion, but it is most convenient to use, rather than an α-system, an $<\alpha_n>$-system. We give the definition for this from [A4].

Definition 2.8.1: Let $\alpha < \omega_1^{CK}$ be a limit ordinal and let $<\alpha_n>$ be a recursive increasing sequence of successor ordinals whose supremum is α.

An *$<\alpha_n>$-system* is a structure $\mathcal{S} = (U,L,P,E,<\leq_\gamma>_{\gamma<\alpha})$ for which (U,L,P,E) is an r.e. scheme, each $\leq_\gamma$ is, uniformly in $\gamma < \alpha$, an r.e. binary relation on L, and the following conditions are satisfied.

(1) Each $\leq_\gamma$ is reflexive and transitive.

(2) If $\gamma < \beta < \alpha$ and $l \leq_\beta m$, then $l \leq_\gamma m$.

(3) If $l \leq_0 m$ then $E(l) \subseteq E(m)$.

(4) If $(u_0,l_0,\ldots u_n,l_n,u_{n+1}) \in P$, if $0 \leq \gamma_k < \ldots < \gamma_2 < \gamma_1 < \alpha_n$ and if $l_n = m_1 \leq_{\gamma_1} m_2 \leq_{\gamma_2} \ldots \leq_{\gamma_{k-1}} m_k$ then there exists l_{n+1} for which $(u_0,l_0,\ldots u_n,l_n,u_{n+1},l_{n+1}) \in P$ and $m_i \leq_{\gamma_i} l_{n+1}$ for each $i = 1,2,\ldots,k$.

(This is the same as the definition of an α-system except that in condition (4) α is replaced by α_n.)

An *$\langle\alpha_n\rangle$-input* for $\mathcal{S}$ is an input (u,l,p) for which each restriction of p to arguments of the form $(u_0,l_0,\ldots u_n,l_n)$ is $\Delta^0_{\alpha_n}$ uniformly in n, and an *$\langle\alpha_n\rangle$-index* for such a p is a recursive index for a corresponding sequence of $\Delta^0_{\alpha_n}$ indices.

Proposition 2.8.2: Every $\langle\alpha_n\rangle$-input for an $\langle\alpha_n\rangle$-system has a conservative Δ^0_α run.

Proof: See [A4]. □

Theorem 2.8.3: Let $\alpha < \omega_1^{CK}$ be a limit ordinal and let $\langle\alpha_n\rangle$ be a recursive increasing sequence of successor ordinals whose supremum is α. Let $\mathfrak{A}$ be an r.e. structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:

(1) The positive existential diagram of $\mathfrak{A}$ is recursive.

(2) R is a recursive set.

(3) The relations $\leq^+_\beta$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which finds, given **a** and

n, some $r \in R - Rcl^+_{\alpha_n}(\mathbf{a})$.

Then R is not intrinsically $\hat{\Sigma}^{0+}_\alpha$.

Proof: We construct, using an $\langle\alpha_n\rangle$-system, an appropriate r.e. structure $\mathfrak{B}$ in which the relation corresponding to R is not a Σ^0_β

set for any $\beta < \alpha$.

Let $U = \{0,1\}$. The intention is that $u_{i+1} = 1$ (respectively 0) should mean, on the true path of the construction, that m is (respectively, is not) in the e^{th} $\Sigma^0_{\alpha_n}$ set, where $i = \langle e,n \rangle$ and m is some number determined by l_i. We will require that $n < i$ so that the $\Delta^0_{\alpha_i}$ oracle at stage i will be able to decide all such questions.

Let L be the set of quintuples (n,f,x,g,D) where f,g are finite one-one partial functions from B to $\mathfrak{A}$, $f \subseteq g$, $x \in R - Rcl_{\alpha_n}(\ ran(f)\ )$, $x,a_n \in ran(g)$ and $b_n \in dom(g)$, and D is (the set of Gödel numbers of) a finite set of atomic sentences $\varphi(\mathbf{b})$ of $L(B)$ for which $\mathbf{b} \in dom(g)$ and $\mathfrak{A} \models \varphi(g(\mathbf{b}))$.

Let P be the set of finite sequences $(u_0, l_0, u_1, l_1, \ldots)$ where $l_i = (i, f_i, x_i, g_i, D_i)$ such that, for each i,

(i) $D_i \subseteq D_{i+1}$

(ii) D_i contains each atomic sentence $\varphi(\mathbf{b})$ of $L(B)$ for which $\varphi(g_i(\mathbf{b}))$ has appeared in the first i stages of the enumeration of the diagram of $\mathfrak{A}$.

(iii) If $u_{i+1} = 0$, then $f_{i+1} = g_i$.

(iv) If $u_{i+1} = 1$, then $f_{i+1} \supseteq f_i$ and $f_{i+1} \circ g_i^{-1}(x_i) \notin R$.

If $l = (n,f,x,g,D)$ we define $E(l) = D$. If $l' = (n',f',x',g',D')$ we define $l \leq_\xi l'$ iff $D \subseteq D'$ and $g \angle^+_\xi g'$.

Lemma 2.8.4: This forms an $\langle \alpha_n \rangle$-system.

Proof: This is clear except for condition (4).

To establish this, suppose that $(u_0,l_0,\ldots,u_n,l_n,u_{n+1}) \in P$, $\alpha_n > \xi_1 > \xi_2 > \ldots > \xi_k \geq 0$ and $l_n = m_1 \subseteq_{\xi_1} m_2 \subseteq_{\xi_2} \ldots \subseteq_{\xi_{k-1}} m_k$ where $l_n = (n,\ f_n,\ x_n,\ g_n,\ D_n)$ and $m_j = (n'_j,\ f'_j,\ x'_j,\ g'_j\ ,\ D'_j)$. Then we have by definition

$$g_n = g'_1 \angle^{+}_{\xi_1} g'_2 \angle^{+}_{\xi_2} \ldots \angle^{+}_{\xi_k} g'_k \quad \text{and} \quad D'_1 \subseteq D'_2 \subseteq \ldots \subseteq D'_k.$$

Let $\varphi(\mathbf{b})$ be the conjunction of the sentences of D'_k. Then, since $m_k \in L$, $\mathfrak{A} \models \varphi(g'_k(\mathbf{b}))$. So, by Lemma 2.3.9, there exists $h \supseteq g_n$ such that $\mathfrak{A} \models \varphi(h(\mathbf{b}))$ and $g'_j \angle^{+}_{\xi_j} h$ for each $j = 1,2,\ldots,k$.

In the case where $u_{n+1} = 0$, we may take the required l_{n+1} to be $(\ n+1,\ g_n,\ x',\ h',\ D')$ where $h' \supseteq h$ and $D' \supseteq D'_k$ are chosen to satisfy the definitions of L and P, and

$$x' \in R - Rcl^{+}_{\alpha_{n+1}}(\ ran(g_n)\ ).$$

In the other case, where $u_{i+1} = 1$, let $ran(h) = \mathbf{a},x_n,\mathbf{b}$ where $ran(f_n) = \mathbf{a}$. Now $x_n \in R - Rcl^{+}_{\alpha_n}(\mathbf{a})$, so there are x' and $\mathbf{b}'$ with $\mathbf{a},x_n,\mathbf{b} \leq^{+}_{\xi_1} \mathbf{a},x',\mathbf{b}'$, $\mathfrak{A} \models \varphi(\mathbf{a},x',\mathbf{b}')$ and $x' \notin R$.

Define $h' : dom(h) \longrightarrow \mathfrak{A}$ by $h' \circ h^{-1}(\mathbf{a},x_n,\mathbf{b}) = \mathbf{a},x',\mathbf{b}'$.

Now choose $h'' \supseteq h'$ and $D' \supseteq D'_k$ to satisfy the definitions of L and P, and $x'' \in R - Rcl^{+}_{\alpha_{n+1}}(ran(g_n))$.

Then we may take the required l_{n+1} to be

$$(\ n+1,\ g_n,\ x'',\ h'',\ D').$$

Thus we have defined an $\langle\alpha_n\rangle$-system. □

Now there are, uniformly in n, $\Delta^0_{\alpha_n}$ functions p_n with $p_n(m) = 1$ (respectively 0) if m is (respectively, is not) in the e^{th} $\Sigma^0_{\alpha_{n'}}$ set, where $n = \langle e,n'\rangle$. (Note that $n' < n$.)

So consider the input $(0,l,p)$ where $l = (0,\emptyset,x,f,\emptyset)$, $x \in R$, $x \in ran(f)$, and $p(u_0,l_0,\ldots,u_i,l_i) = p_i(m)$ where $l_i = (i,f_i,x_i,g_i,D_i)$ and $m = g_i^{-1}(x_i)$.

By Proposition 2.8.2, this input has a conservative Δ^0_α run $(u_0,l_0,\ldots)$. In this run let $l_i = (i,f_i,x_i,g_i,D_i)$.

From the construction we see that $\{f_i\}$ form a chain, and $f = \bigcup f_i : \mathfrak{B} \cong \mathfrak{A}$, where $\mathfrak{B}$ is the r.e. structure whose diagram $\bigcup D_i$ is enumerated in the course of the construction.

Also we see that at the n^{th} level in this chain we have ensured that $f^{-1}(R)$ is not the e^{th} $\Sigma^0_{\alpha_{n'}}$ set, where $n = \langle e,n'\rangle$. Since $lim\ \alpha_n = \alpha$, this shows that $f^{-1}(R)$ is not a $\hat{\Sigma}^0_\alpha$ set. ■

Theorem 2.8.5: Let $\alpha < \omega_1^{CK}$ be a limit ordinal and let $\langle\alpha_n\rangle$ be a recursive increasing sequence of successor ordinals whose supremum is α. Let $\mathfrak{A}$ be an r.e. structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:

(1) The positive existential diagram of $\mathfrak{A}$ is recursive.

(2) R is a recursive set.

(3) The relations $\leq_\beta$ are uniformly r.e. for $\beta < \alpha$.

(4) There is a recursive procedure which determines for each x, n and $\mathbf{a}$ whether $x \in Rcl^+_{\alpha_n}(\mathbf{a})$.

Then R is intrinsically $\hat{\Sigma}^{0+}_\alpha$ iff it is intrinsically Σ^{0+}_β for some

$\beta < \alpha$.

Proof: If R is not intrinsically $\hat{\Sigma}_{\alpha}^{0+}$ it is certainly not intrinsically Σ_{β}^{0+} for any $\beta < \alpha$.

Conversely, if R is intrinsically $\hat{\Sigma}_{\alpha}^{0+}$, then by Theorem 2.8.3 and condition (4), for some **a** and n we must have $R = Rcl_{\alpha_n}(\mathbf{a})$. But then by Propositions 2.4.5 and 2.2.5, R is intrinsically $\Sigma_{\alpha_n}^{0+}$. ■

Of course, there are corresponding results for recursive structures.

CHAPTER 3. EXAMPLES IN LINEAR ORDERINGS.

§3.1 Recursive Linear Orderings.

We will consider various relations on recursive linear orderings, $\mathfrak{A} = (A,<)$. Since the order relation is total, a linear ordering is recursive iff it is r.e. Consequently we consider recursive linear orderings here.

The isomorphism type of a linear ordering is called its *order type*. The order type of the rationals is denoted by η. The order type of a recursive well ordering is a constructive ordinal. The order type of the natural numbers is ω, and of the integers, $\omega^* + \omega$. If φ is any linear ordering then φ^* denotes the reverse linear ordering.

We will abuse notation by referring to order types rather than specific linear orderings, as the order types provide more convenient names. Of course the properties we are interested in depend only on the order types involved. In the case of well orderings we will use ordinals to refer both to intervals and specific points within intervals (since an ordinal is the set of its predecessors).

A *section* in a linear ordering $\mathfrak{A}$ is a set of one of the following forms, or the corresponding subordering:

$\{ x \in A : x < b \}$, $\{ x \in A : a < x < b \}$, or $\{ x \in A : a < x \}$ for $a,b \in A$.

Since $\mathbf{a} \leq_0 \mathbf{b}$ means that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier-free formulae, this happens exactly when the elements of **a** correspond in order to the elements of **b**. In this case there is a correspondence between the sections of A determined by **a** and by **b**.

Proposition 3.1.1: For linear orderings A and B, we have $(A,\mathbf{a}) \le_\alpha (B,\mathbf{b})$ iff $(A,\mathbf{a}) \le_0 (B,\mathbf{b})$ and $A' \le_\alpha B'$ for each pair of corresponding sections.

Proof: By induction, since $(A,\mathbf{a}) \le_\alpha (B,\mathbf{b})$ iff for all $\beta < \alpha$ and all **d** there is **c** such that $(A,\mathbf{a},\mathbf{c}) \ge_\beta (B,\mathbf{b},\mathbf{d})$. □

Thus it suffices to apply the following restatement of the definition of $\le_\alpha$ for order types.

Proposition 3.1.2: For $\alpha \ge 1$ and order types φ, ψ we have $\psi \le_\alpha \varphi$ $\cancel{\varphi \le_\alpha \psi}$ iff for every $\beta < \alpha$ and every choice of $\varphi_0, \varphi_1, \ldots, \varphi_k$ for which

$$\varphi = \varphi_0 + 1 + \varphi_1 + 1 + \ldots + 1 + \varphi_k,$$

there exist $\psi_0, \psi_1, \ldots, \psi_k$ such that

$$\psi = \psi_0 + 1 + \psi_1 + 1 + \ldots + 1 + \psi_k$$

and $\psi_i \ge_\beta \varphi_i$ for each $i = 0,1,\ldots,k$. □

Note that $\varphi \le_0 \psi$ for any order types φ, ψ. Also $\varphi \le_1 \psi$ iff φ contains infinitely many points or ψ is finite and φ contains at least as many points as ψ.

§3.2 Recursive Well-orderings.

The following classical result is given in [Rs] as Theorem 3.46.

Proposition 3.2.1: (Cantor Normal Form) Every ordinal has a unique representation as a finite sum

$$\alpha = \sum_{\xi} \omega^{\xi} m_{\xi}$$

where each m_{ξ} is a natural number and the powers of ω occur in order of decreasing size. □

Definition 3.2.2: Let $\alpha_{\gamma} = \sum_{\xi \geq \gamma} \omega^{\xi} m_{\xi}$ and $\alpha_{(\gamma)} = \sum_{\xi < \gamma} \omega^{\xi} m_{\xi}$.

So $\alpha = \alpha_{\gamma} + \alpha_{(\gamma)}$, where α_{γ} is the nearest γ^{th} limit point to (the left of) α.

The following result giving conditions for the back and forth relations for well orderings appeared in [A2], but no proof has been published, so we give here our own proof. In [A2] the clause (Ic) was omitted inadvertently.

Proposition 3.2.3: If $\alpha = \sum_{\xi} \omega^{\xi} m_{\xi}$ and $\beta = \sum_{\xi} \omega^{\xi} n_{\xi}$ are order types of recursive well orderings in Cantor Normal form, then:

(I) $\alpha \leq_{2\gamma+1} \beta$ iff (a) $\alpha = \beta$, or

(b) $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\beta_\gamma \neq 0$ and $\alpha_\gamma \geq min\ \{\ \beta_\gamma,\ \omega^{\gamma+1}\}$, or

(c) $\gamma = 0$ and $\beta = 0$.

(II) $\alpha \leq_{2\gamma+2} \beta$ iff (a) $\alpha = \beta$, or

(b) $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \geq n_\gamma$, $\alpha_{\gamma+1} \neq 0$ and $\beta_{\gamma+1} \neq 0$.

(III) For limit ordinals γ, $\alpha \leq_\gamma \beta$ iff (a) $\alpha = \beta$, or

(b) $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\alpha_\gamma \neq 0$ and $\beta_\gamma \neq 0$, or

(c) $\gamma = 0$.

Proof: By induction on γ, using Proposition 3.1.2. The base case (IIIc) holds by definition. The proof is in two parts. In part A, we show that if α and β satisfy the given condition then given any finite partition of $\overset{\beta}{\cancel{\alpha}}$ we can find a corresponding partition of $\overset{\alpha}{\cancel{\beta}}$ so that the corresponding sections have inductively the right relationship. Conversely in part B we show that if α and β do not satisfy the condition then we can find a partition of $\overset{\beta}{\cancel{\alpha}}$ for which no partition of $\overset{\alpha}{\cancel{\beta}}$ gives the corresponding sections the right relationship.

A.

The cases (AIa), (AIIa) and (AIIIa) are dealt with by observing that if $\alpha = \beta$, then $\alpha \leq_\delta \beta$ for every δ.

(AIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\beta_\gamma \neq 0$ and $\alpha_\gamma \geq min\ \{\ \beta_\gamma,\ \omega^{\gamma+1}\}$.

Since $\alpha_{(\gamma)} = \beta_{(\gamma)}$, any point in the interval $\beta_{(\gamma)}$ can be mapped to the corresponding point in the interval $\alpha_{(\gamma)}$.

For each point σ in the interval β_γ, introduce another point at σ_γ. These induce finitely many intervals in β_γ, each of type $\omega^\gamma\varphi$ for some ordinal φ. Since $\omega^\gamma\varphi \leq_{2\gamma} \omega^\gamma\theta$ for any ordinal θ, by the induction hypothesis, and since either α_γ is at least as large as β_γ or α_γ contains infinitely many intervals of length ω^γ, we can find appropriate points to map each σ_γ to in α_γ, say $\omega^\gamma\tau$. Then map each σ to $\omega^\gamma\tau + \sigma_{(\gamma)}$. This ensures that all intervals and tails of length $< \omega^\gamma$ correspond to intervals or tails of the same length.

(AIc) Suppose $\gamma = 0$ and $\beta = 0$.
Since we cannot divide β at points, $\alpha \leq_1 0$ iff $0 \leq_0 \alpha$, which is true.

(AIIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \geq n_\gamma$, $\alpha_{\gamma+1} \neq 0$ and $\beta_{\gamma+1} \neq 0$. Then

$$\beta = \omega^{\gamma+1}\beta' + \omega^\gamma n_\gamma + \beta_{(\gamma)} \text{ and}$$

$$\alpha = \omega^{\gamma+1}\alpha' + \omega^\gamma(m_\gamma - n_\gamma) + \omega^\gamma n_\gamma + \alpha_{(\gamma)}$$

Any point in the final $\omega^\gamma n_\gamma + \beta_{(\gamma)}$ can be mapped to the corresponding point in that tail of α.

For each point σ in the interval $\omega^{\gamma+1}\beta'$, introduce another point at σ_γ. These induce finitely many intervals in $\omega^{\gamma+1}\beta'$, each of type $\omega^\gamma\varphi$ for some non-zero ordinal φ. Map the k^{th} σ_γ to $\omega^\gamma k$ in $\omega^{\gamma+1}\alpha'$. Then the final interval in $\omega^{\gamma+1}\beta'$, of type $\omega^{\gamma+1}\theta$, corresponds to an interval of type $\omega^{\gamma+1}\alpha' + \omega^\gamma(m_\gamma - n_\gamma)$. This is appropriate since, by the induction hypothesis,

$$\omega^{\gamma}\varphi \leq_{2\gamma+1} \omega^{\gamma} \text{ and } \omega^{\gamma+1}\theta \leq_{2\gamma+1} \omega^{\gamma+1}\alpha' + \omega^{\gamma}(m_{\gamma}-n_{\gamma}).$$

Then map each σ associated with the k^{th} σ_{γ} to $\omega^{\gamma}k + \sigma_{(\gamma)}$.

(AIIIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\alpha_{\gamma} \neq 0$ and $\beta_{\gamma} \neq 0$.

Then we see that, by the induction hypothesis, for each $\delta < \gamma$, $\alpha \leq_{2\delta+2} \beta$. So, for any subdivision of β, we can find a subdivision of α in the $\geq_{2\delta+1}$ relation with it.

B.

(I) Suppose $\alpha \neq \beta$ and

$\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $\beta_{\gamma} = 0$ or ($\alpha_{\gamma} < \beta_{\gamma}$ and $\alpha_{\gamma} < \omega^{\gamma+1}$) and

$\gamma \neq 0$ or $\beta \neq 0$.

We deal with the various possibilities case by case.

(BIa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

If γ is a limit ordinal this implies *not* $\alpha \leq_{\gamma} \beta$, so *not* $\alpha \leq_{\gamma+1} \beta$.

If $\gamma = \delta+1$, we have $\alpha_{(\delta)} \neq \beta_{(\delta)}$ or $m_{\delta} \neq n_{\delta}$.

If $\alpha_{(\delta)} \neq \beta_{(\delta)}$ or $m_{\delta} < n_{\delta}$, we have *not* $\alpha \leq_{2\delta+2} \beta$.

So suppose $n_{\delta} < m_{\delta}$.

We have $\beta = \beta_{\delta+1} + \omega^{\delta}n_{\delta} + \beta_{(\delta)}$ and $\alpha = \alpha_{\delta+1} + \omega^{\delta}m_{\delta} + \alpha_{(\delta)}$.

Choose points at β_{δ} and $\beta_{\delta+1}$. By induction we must map β_{δ} to α_{δ}.

If we map $\beta_{\delta+1}$ to any point less than $\alpha_{\delta+1} + \omega^{\delta}(m_{\delta}-n_{\delta})$ we have the interval $\omega^{\delta}n_{\delta}$ corresponding to one of length $\geq \omega^{\delta}(n_{\delta}+1)$, which does not satisfy the inductive requirement in (II) for $\leq_{2\delta+2}$.

So we can only map $\beta_{\delta+1}$ to some point $\geq \alpha_{\delta+1} + \omega^{\delta}(m_{\delta}-n_{\delta})$. But then the interval $\beta_{\delta+1}$ corresponds to some interval $\geq \alpha_{\delta+1} + \omega^{\delta}(m_{\delta}-n_{\delta})$, which again fails to satisfy the inductive requirement.

(BIb) Suppose $\gamma \neq 0$, $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\beta_{\gamma} = 0$ and $\alpha_{\gamma} \neq 0$.

Then $\beta = \beta_{(\gamma)}$ and $\alpha = \alpha_{\gamma} + \alpha_{(\gamma)}$.

But, by (II) or (III), *not* $\beta_{(\gamma)} \leq_{2\gamma} \alpha_{\gamma} + \alpha_{(\gamma)}$.

(BIc) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\beta_{\gamma} \neq 0$, $\alpha_{\gamma} < \beta_{\gamma}$ and $\alpha_{\gamma} < \omega^{\gamma+1}$.

Then $\alpha = \omega^{\gamma}m + \alpha_{(\gamma)}$ and $\beta = \omega^{\gamma}(m+1+\theta) + \beta_{(\gamma)}$ for some ordinal θ. Choose points in β at ω^{γ}, $\omega^{\gamma}2$, ..., $\omega^{\gamma}(m+1)$. Then whatever points we choose in α, we must have an interval of length ω^{γ} corresponding to one of length $< \omega^{\gamma}$, which contradicts the inductive requirement of (II) or (III).

(II) Suppose $\alpha \neq \beta$ and

$$\alpha_{(\gamma)} \neq \beta_{(\gamma)} \text{ or } m_{\gamma} < n_{\gamma} \text{ or } \alpha_{\gamma+1} = 0 \text{ or } \beta_{\gamma+1} = 0.$$

(BIIa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

Then by (I) *not* $\alpha \leq_{2\gamma+1} \beta$.

(BIIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $m_{\gamma} < n_{\gamma}$.

Then $\alpha = \alpha_{\gamma+1} + \omega^{\gamma}m + \alpha_{(\gamma)}$ and $\beta = \beta_{\gamma+1} + \omega^{\gamma}(m+1+k) + \beta_{(\gamma)}$ for some k.

Choose points in β at $\beta_{\gamma+1}$, $\beta_{\gamma+1} + \omega^{\gamma}$, $\beta_{\gamma+1} + \omega^{\gamma}2, \ldots$, $\beta_{\gamma+1} + \omega^{\gamma}(m+k)$, and β_{γ}. By induction β_{γ} must map to α_{γ}.

If we map $\beta_{\gamma+1}$ to a point $\geq \alpha_{\gamma+1}$, then some interval of length ω^{γ}

corresponds to an interval of length $< \omega^{\gamma}$, contradicting (I).

Note that if $\gamma = 0$, it is not possible to map β_1 to a point $\geq \alpha_1$.

If we map $\beta_{\gamma+1}$ to a point $< \alpha_{\gamma+1}$ then an interval of length ω^{γ} corresponds to an interval of length $\geq \omega^{\gamma+1}$, contradicting (I).

(BIIc) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_{\gamma} \geq n_{\gamma}$, $\alpha_{\gamma+1} = 0$ and $\beta_{\gamma+1} \neq 0$.

Then $\beta = \beta_{\gamma+1} + \omega^{\gamma} n_{\gamma} + \beta_{(\gamma)}$ and $\alpha = \omega^{\gamma} m_{\gamma} + \alpha_{(\gamma)}$.

Choose points in $\beta_{\gamma+1}$ at ω^{γ}, $\omega^{\gamma}2, \ldots$, $\omega^{\gamma}(m+1)$. Then an interval of length ω^{γ} must correspond to an interval of length $< \omega^{\gamma}$, contradicting (I).

(BIId) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_{\gamma} \geq n_{\gamma}$, $\beta_{\gamma+1} = 0$ and ($\alpha_{\gamma+1} \neq 0$ or $m_{\gamma} > n_{\gamma}$).

Then $\beta = \omega^{\gamma} n_{\gamma} + \beta_{(\gamma)}$ and $\alpha = \omega^{\gamma}(n_{\gamma}+1+\theta) + \alpha_{(\gamma)}$ for some ordinal θ.

From (I) we have *not* $\beta \leq_{2\gamma+1} \alpha$.

(III) Suppose $\alpha \neq \beta$ and

$\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $\alpha_{\gamma} = 0$ or $\beta_{\gamma} = 0$.

(BIIIa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

Then for some $\delta < \gamma$, $\alpha_{(\delta)} \neq \beta_{(\delta)}$, so *not* $\alpha \leq_{2\delta+2} \beta$.

(BIIIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\alpha_{\gamma} = 0$ and $\beta_{\gamma} \neq 0$

or $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $\alpha_{\gamma} \neq 0$ and $\beta_{\gamma} = 0$.

Then for some $\delta < \gamma$, $\alpha_{\delta} = 0$ (respectively $\beta_{\delta} = 0$) while $\beta_{\delta} \neq 0$ (respectively $\alpha_{\delta} \neq 0$). So *not* $\alpha \leq_{2\delta+2} \beta$.

This completes all the cases of the induction. □

From this result we see that for the relations $\{\leq_\gamma\}_{\gamma<\alpha}$ to be uniformly recursive in a well ordering it suffices that we be able to identify the Cantor Normal form of any point in the ordering, for then we can calculate how big the interval between two points is.

So given a notation for a constructive ordinal $\xi = |x|_{\mathcal{O}}$, define a *Cantor notation* as follows.

Definition 3.2.4: Suppose $m_i \in \mathbb{N}$, $a_i \in \mathcal{O}$ and $a_{i+1} <_{\mathcal{O}} a_i <_{\mathcal{O}} x$ for each i.
Then $\langle a_1, m_1, \ldots, a_k, m_k \rangle$ is a *Cantor notation* for $\sum_i \omega^{|a_i|} m_i$.

Proposition 3.2.5: For any constructive ordinals σ, α there is a recursive linear ordering of order type σ for which the relations $\{\leq_\gamma\}_{\gamma<\alpha}$ are uniformly recursive and the existential diagram is recursive.

Proof: Take an appropriate set of Cantor notations ordered in the natural way. □

We will consider L_α, the set of α^{th} limit points in recursive well orderings.

Proposition 3.2.6: In any recursive well ordering $(\sigma,<)$ the set of α^{th} limit points, L_α, is formally $\Pi^0_{2\alpha}$. If L_α is infinite, then it is not formally $\Sigma^0_{2\alpha}$.

Proof: We have the following $\Pi_{2\alpha}$ formulae

$$x \in L_1 \longleftrightarrow \forall y\ \exists z\ [\ y < x \rightarrow y < z < x\ ]$$

$$x \in L_{\gamma+1} \longleftrightarrow \forall y\ \exists z\ [(y < x\ \&\ y \in L_\gamma) \rightarrow (y < z < x\ \&\ z \in L_\gamma)]$$

$$x \in L_\gamma \longleftrightarrow \bigwedge_n [\ x \in L_{\gamma_n}\ ] \text{ if } \gamma = lim\ \gamma_n.$$

If L_α is infinite then $\sigma \geq \omega^{\alpha+1}$. Let $R = L_\alpha$. Given **a**, we want to find $r \in R - Rcl_\alpha$.

Let $r = max\ \{\ a_i : a_i \in \mathbf{a} \text{ and } a_i < \omega^{\alpha+1}\ \} + \omega^\alpha$.

Now suppose that **b** and $2\beta+1 < \alpha$ are given. Let $r' = r + \omega^\beta$ and

$$b'_i = \begin{cases} b_i + \omega^\beta \text{ if } r \leq b_i < r + \omega^{\beta+1} \\ b_i \text{ otherwise} \end{cases}$$

Then, since $\omega^\alpha \leq_{2\beta+1} \omega^\alpha + \omega^\beta$ and no other interval changes size in the correspondence, we have $\mathbf{a},r,\mathbf{b} \leq_{2\beta+1} \mathbf{a},r',\mathbf{b}'$. So $r \in R - Rcl_\alpha$.

□

We may apply Theorem 2.7.1, to conclude:

Proposition 3.2.7: Let $\alpha > 0$ and $\sigma \geq \omega^{\alpha+1}$ be constructive ordinals. Then there is a recursive well ordering of order type σ in which the set of α^{th} limit points is not a $\Sigma^0_{2\alpha}$ set.

Proof: By Proposition 3.2.5, conditions (1), (2) and (3) of Theorem 2.7.1 are satisfied. By the proof of Proposition 3.2.6, condition (4) is satisfied. □

§3.3 Some Other Recursive Linear Orderings.

Any interval in a recursive well ordering is intrinsically recursive as it is defined by a quantifier-free formula involving two parameters. Thus $x \in S \longleftrightarrow a \leq x < b$, where a is the least element of S and b is the least element of $\{ y : \forall x \in S\ (x < y) \}$. (Or $x \in S \longleftrightarrow a \leq x$.) This may not be the case in other linear orderings, where not every interval is actually a section.

For example there is a theorem of Tennenbaum ([Rs], Theorem 16.54) which proves the existence of a recursive linear ordering of order type $\omega + \omega^*$ in which there is no infinite r.e. ascending or descending sequence. This implies the following.

Proposition 3.3.1: The set $L = \{x : \bigvee_k \forall z\ \mathit{not}\ (z_1 < z_2 < \ldots < z_k < x)\}$ in a recursive linear ordering of type $\omega + \omega^*$ is intrinsically Δ^0_2 but not intrinsically Σ^0_1 or Π^0_1.

Proof: The above formula shows that L is formally Σ^0_2. Since the complement of L has a similar Σ_2 definition, L is also formally Π^0_2. Tennenbaum's theorem provides a recursive linear ordering in which L is neither r.e. nor co-r.e. □

If we consider linear orderings of type $\omega^\alpha + (\omega^\alpha)^*$, we might initially think of generalizing the definition of L above thus.

Proposition 3.3.2: Define the formulae φ_γ as follows:

$$\varphi_1(x,y) \longleftrightarrow \bigvee\!\!\!\bigvee_k \forall z \; not \; (x = z_1 < z_2 < \ldots < z_k = y)$$
$$\varphi_{\gamma+1}(x,y) \longleftrightarrow \bigvee\!\!\!\bigvee_k \forall z \; (x = z_1 < z_2 < \ldots < z_k = y) \longrightarrow \bigvee_i \varphi_\gamma(z_i, z_{i+1})$$
$$\varphi_\gamma(x,y) \longleftrightarrow \bigvee\!\!\!\bigvee_{\beta<\gamma} \varphi_\beta(x,y) \qquad \text{for limit ordinals } \gamma.$$

Then in any linear ordering $\varphi_\gamma(x,y)$ implies that there is no suborder of type ω^γ in the interval (x,y). Furthermore the converse is true in a well ordering.

Proof: By induction on γ. □

So in $\mathfrak{A} = (\, \omega^\alpha + (\omega^\alpha)^*, < \,)$, the 'left half' L is formally $\Sigma^0_{2\alpha}$ and by symmetry formally $\Pi^0_{2\alpha}$. This suggests that we might try to show that L is, for example, not formally Σ^0_β for any $\beta < 2\alpha$.

A quick analysis of the back and forth relations in this case shows that we cannot have thought carefully enough about how to define L:

Proposition 3.3.3: If α, β are ordinals then

(i) *not* $\alpha \leq_2 \beta^*$ unless $\alpha = \beta^*$;

(ii) *not* $\alpha \leq_2 \beta + \beta^*$ unless $\alpha = \beta + \beta^*$.

Proof: (i) If α is finite, $\alpha \leq_2 \beta^*$ iff $\alpha = \beta$.
(ii) If α is finite, $\alpha \leq_2 \beta + \beta^*$ iff $\alpha = \beta + \beta^*$.
So in both cases suppose $\alpha = \omega\alpha' + k$ where k is finite. Choose the

last $k+1$ points in β^*. (If this is not possible we are finished because a finite interval is not $\leq_1$ an infinite interval.) Then one of the empty intervals so formed must correspond to an infinite interval in α, which is not possible under the $\leq_1$ relation. □

In fact, L has a Σ_3 definition in any $\mathfrak{A}$ = $(\omega^2\theta + (\omega^2\theta)^*, <)$, where θ is a constructive ordinal.
x is in L iff for some k, x is k points to the right of a left limit point.

Proposition 3.3.4: In a recursive linear ordering $\mathfrak{A}$ = $(\omega^2\theta + (\omega^2\theta)^*, <)$ the set L consisting of the initial $\omega^2\theta$ is formally Δ^0_3 but not formally Σ^0_2 or Π^0_2.

Proof: We have the Σ_3 definition

$$x \in L \;\longleftrightarrow\; \bigvee\!\!\bigvee_k \exists y\, \forall z\, \exists w\, [\; y_1 < y_2 < \ldots < y_k = x \;\;\&\;\; \bigwedge_{i<k} \textit{not}\; (y_i < z < y_{i+1}) \;\;\&\;\; (z < y_1) \longrightarrow (z < w < y_1) \;]$$

Since the complement of L has a similar definition, L is also formally Π^0_3. We will show that L is not formally Σ^0_2.

Suppose $\mathbf{a}$ is given. Let $a_L = \max \{\, a \in \mathbf{a} : a \in L \,\}$.
Choose $r = a_L + \omega$. Then, given $\mathbf{b}$, let $b_R = \min \{\, b \in \mathbf{a},\mathbf{b} : b \notin L \,\}$ and let $\mathbf{b}_m$ be those points in $\mathbf{b}$ which are in L but to the right of r. Then map $\mathbf{b}_m$ in order preserving fashion to the points

immediately to the left of b_R and let r' be the point immediately to the left of them. Leave the rest of the **b** unchanged.

Then all intervals have been compressed except for the one with upper limit r. It is of type ω, which is $\leq_1$ any linear ordering.

So $\mathbf{a},r,\mathbf{b} \leq_1 \mathbf{a},r',\mathbf{b'}$ and so $r \in R - Rcl_2(\mathbf{a})$.

Similarly by symmetry L is not formally Π^0_2. $\square$

In order to find situations where the φ_γ formulae from Proposition 3.3.2 are appropriate, we consider linear orderings of type $\omega^\rho(1+\eta)$, $\omega^\rho(1+\omega^*)$ and $\omega^\rho(1+\omega^*+\omega)$.

The following is quite similar to Proposition 3.2.3 above.

Proposition 3.3.5: If $\alpha = \sum_\xi \omega^\xi m_\xi$ and $\beta = \sum_\xi \omega^\xi n_\xi$ are order types of recursive well orderings in Cantor Normal form and $\alpha,\beta \leq \omega^\rho$, then:

(I) $\omega^\rho(1+\eta) + \alpha \leq_{2\gamma+1} \beta$ iff

(a) $\alpha_{(\gamma)} = \beta_{(\gamma)}$, and $\beta_\gamma \neq 0$ or

(b) $\gamma = 0$ and $\beta = 0$.

(II) $\omega^\rho(1+\eta) + \alpha \leq_{2\gamma+2} \beta$ iff $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \geq n_\gamma$ and $\beta_{\gamma+1} \neq 0$.

(III) For limit ordinals γ, $\omega^\rho(1+\eta) + \alpha \leq_\gamma \beta$ iff

(a) $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $\beta_\gamma \neq 0$, or

(c) $\gamma = 0$.

(IV) $\beta \leq_{2\gamma+1} \omega^{\rho}(1+\eta) + \alpha$ iff $\alpha_{(\gamma)} = \beta_{(\gamma)}$, and $\beta_{\gamma+1} \neq 0$.

(V) $\beta \leq_{2\gamma+2} \omega^{\rho}(1+\eta) + \alpha$ iff $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \leq n_\gamma$ and $\beta_{\gamma+1} \neq 0$.

(VI) For limit ordinals γ, $\beta \leq_\gamma \omega^{\rho}(1+\eta) + \alpha$ iff

(a) $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $\beta_\gamma \neq 0$, or

(c) $\gamma = 0$.

Proof: By induction on γ, in two parts.

A.

(AI) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, and $\beta_\gamma \neq 0$.
For each point in β_γ, σ, introduce σ_γ. These induce finitely many intervals of type $\omega^\gamma\theta$ ($\theta \neq 0$). Since $\omega^\gamma\theta \leq_{2\gamma} \omega^{\rho}(1+\eta) + \omega^\gamma\varphi$ for any ordinal φ, by (V) or (VI), these points can be matched.

(AII) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \geq n_\gamma$ and $\beta_{\gamma+1} \neq 0$.
Map points in $\omega^\gamma n_\gamma + \beta_{(\gamma)}$ to the corresponding positions at the end of α. Map each point in $\beta_{\gamma+1}$ to the corresponding point in the initial ω^{ρ}. Since by (IV), $\omega^{\gamma+1}\theta \leq_{2\gamma+1} \omega^{\rho}(1+\eta) + \omega^\gamma\varphi$, all the intervals match appropriately.

(AIII) and (AVI) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $\beta_\gamma \neq 0$.
Then we have $\omega^{\rho}(1+\eta) + \alpha \leq_\delta \beta$, for each $\delta < \gamma$.

(AIV) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, and $\beta_{\gamma+1} \neq 0$.

Map any points in the final $\alpha_{(\gamma)}$ to the corresponding point in $\beta_{(\gamma)}$. For each other point σ, find the next γ^{th} limit point to its left, σ'. These induce finitely many intervals of type $\omega^{\rho}(1+\eta) + \omega^{\gamma}\theta$ or $\omega^{\gamma}\theta$. Since $\omega^{\rho}(1+\eta) + \omega^{\gamma}\theta \leq_{2\gamma} \omega^{\gamma}\varphi$ for any ordinal φ, these σ' can be mapped to γ^{th} limit points in $\beta_{\gamma+1}$.

(AV) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_{\gamma} \leq n_{\gamma}$ and $\beta_{\gamma+1} \neq 0$.

Points in the final $\omega^{\gamma}m_{\gamma} + \alpha_{(\gamma)}$ can be mapped to the corresponding points at the end of β. For other points σ, as before find the next γ^{th} limit points to their left, σ', and map them to γ^{th} limit points in $\beta_{\gamma+1}$.

B.

(I) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $\beta_{\gamma} = 0$ and

$\gamma \neq o$ or $\beta \neq o$.

(BIa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

If γ is a limit ordinal this implies *not* $\omega^{\rho}(1+\eta) + \alpha \leq_{\gamma} \beta$.

If $\gamma = \delta+1$, we have $\alpha_{(\delta)} \neq \beta_{(\delta)}$ or $m_{\delta} \neq n_{\delta}$.

If $\alpha_{(\delta)} \neq \beta_{(\delta)}$ or $m_{\delta} < n_{\delta}$, we have *not* $\omega^{\rho}(1+\eta) + \alpha \leq_{2\delta+2} \beta$.

So suppose $n_{\delta} < m_{\delta}$.

We have $\beta = \beta_{\delta+1} + \omega^{\delta}n_{\delta} + \beta_{(\delta)}$ and $\alpha = \alpha_{\delta+1} + \omega^{\delta}m_{\delta} + \alpha_{(\delta)}$.

Choose points at β_{δ} and $\beta_{\delta+1}$. By induction we must map β_{δ} to α_{δ}.

If we map $\beta_{\delta+1}$ to any point less than $\alpha_{\delta+1} + \omega^{\delta}(m_{\delta}-n_{\delta})$ we have the interval $\omega^{\delta}n_{\delta}$ corresponding to one of length $\geq \omega^{\delta}(n_{\delta}+1)$, which does not satisfy the requirement in (II) of Proposition 3.2.3 for $\leq_{2\delta+2}$.

So we can only map $\beta_{\delta+1}$ to some point $\geq \alpha_{\delta+1} + \omega^{\delta}(m_{\delta}-n_{\delta})$. But then the interval $\beta_{\delta+1}$ corresponds to some interval $\geq \omega^{\rho}(1+\eta) + \alpha_{\delta+1} + \omega^{\delta}(m_{\delta}-n_{\delta})$, which fails to satisfy the inductive requirement in (V).

(BIb) Suppose $\gamma \neq 0$, $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $\beta_{\gamma} = 0$.

Then $\beta = \beta_{(\gamma)}$. But, by (V) or (VI),

not $\beta_{(\gamma)} \leq_{2\gamma} \omega^{\rho}(1+\eta) + \alpha_{\gamma} + \alpha_{(\gamma)}$.

(II) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $m_{\gamma} < n_{\gamma}$ or $\beta_{\gamma+1} = 0$.

(BIIa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

Then by (I) *not* $\omega^{\rho}(1+\eta) + \alpha \leq_{2\gamma+1} \beta$.

(BIIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $m_{\gamma} < n_{\gamma}$.

Then $\alpha = \alpha_{\gamma+1} + \omega^{\gamma}m + \alpha_{(\gamma)}$ and $\beta = \beta_{\gamma+1} + \omega^{\gamma}(m+1+k) + \beta_{(\gamma)}$ for some k.

Choose points in β at $\beta_{\gamma+1}$, $\beta_{\gamma+1} + \omega^{\gamma}$, $\beta_{\gamma+1} + \omega^{\gamma}2, \ldots,$ $\beta_{\gamma+1} + \omega^{\gamma}(m+k)$, and β_{γ}. By induction β_{γ} must map to α_{γ}.

If we map $\beta_{\gamma+1}$ to a point $\geq \alpha_{\gamma+1}$, then some interval of length ω^{γ} corresponds to an interval of length $< \omega^{\gamma}$, contradicting (I) in Proposition 3.2.3.

Note that if $\gamma = 0$, it is not possible to map β_1 to a point $\geq \alpha_1$.

If we map $\beta_{\gamma+1}$ to a point $< \alpha_{\gamma+1}$ then an interval of length ω^{γ}

corresponds to an interval of length $\geq \omega^{\gamma+1}$, contradicting (I) in Proposition 3.2.3 or (IV).

(BIIc) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \geq n_\gamma$ and $\beta_{\gamma+1} = 0$. From (IV) we have *not* $\beta \leq_{2\gamma+1} \omega^\rho(1+\eta) + \alpha$.

(III) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $\beta_\gamma = 0$.

(BIIIa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

Then for some $\delta < \gamma$, $\alpha_{(\delta)} \neq \beta_{(\delta)}$, so *not* $\omega^\rho(1+\eta) + \alpha \leq_{2\delta+2} \beta$.

(BIIIb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $\beta_\gamma = 0$.

Then for some $\delta < \gamma$, $\beta_\delta = 0$. So *not* $\omega^\rho(1+\eta) + \alpha \leq_{2\delta+2} \beta$.

(IV) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $\beta_{\gamma+1} = 0$.

(BIVa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

If γ is a limit ordinal this implies *not* $\beta \leq_\gamma \omega^\rho(1+\eta) + \alpha$.

If $\gamma = \delta+1$, we have $\alpha_{(\delta)} \neq \beta_{(\delta)}$ or $m_\delta \neq n_\delta$.

If $\alpha_{(\delta)} \neq \beta_{(\delta)}$ or $m_\delta > n_\delta$, we have *not* $\beta \leq_{2\delta+2} \omega^\rho(1+\eta) + \alpha$.

So suppose $n_\delta > m_\delta$.

We have $\beta = \beta_{\delta+1} + \omega^\delta n_\delta + \beta_{(\delta)}$ and $\alpha = \alpha_{\delta+1} + \omega^\delta m_\delta + \alpha_{(\delta)}$.

Choose points at α_δ and $\alpha_{\delta+1}$. By induction we must map α_δ to β_δ.

If we map $\alpha_{\delta+1}$ to any point less than $\beta_{\delta+1} + \omega^\delta(n_\delta - m_\delta)$ we have the interval $\omega^\delta m_\delta$ corresponding to one of length $\geq \omega^\delta(m_\delta+1)$, which does not satisfy the requirement in (II) of Proposition 3.2.3 for

$\leq_{2\delta+2}$.

So we can only map $\alpha_{\delta+1}$ to some point $\geq \beta_{\delta+1} + \omega^{\delta}(n_{\delta}-m_{\delta})$. But then the interval $\omega^{\rho}(1+\eta) + \alpha_{\delta+1}$ corresponds to some interval $\geq \beta_{\delta+1} + \omega^{\delta}(n_{\delta}-m_{\delta})$, which fails to satisfy the inductive requirement in (II).

(BIVb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $\beta_{\gamma+1} = 0$.

Then $\beta = \omega^{\gamma}n_{\gamma} + \beta_{(\gamma)}$. Choose $n_{\gamma}+1$ initial points of different copies of ω^{ρ}. Then at least one of the intervals so formed will have to be mapped to an interval of length $< \omega^{\gamma}$. But, by (II) or (III), *not* $\omega^{\rho}(1+\eta) \leq_{2\gamma} \theta$ if θ is of length $< \omega^{\gamma}$.

(V) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$ or $m_{\gamma} > n_{\gamma}$ or $\beta_{\gamma+1} = 0$.

(BVa) Suppose $\alpha_{(\gamma)} \neq \beta_{(\gamma)}$.

Then by (IV) *not* $\beta \leq_{2\gamma+1} \omega^{\rho}(1+\eta) + \alpha$.

(BVb) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$ and $m_{\gamma} > n_{\gamma}$.

Then $\beta = \beta_{\gamma+1} + \omega^{\gamma}n + \beta_{(\gamma)}$ and $\alpha = \alpha_{\gamma+1} + \omega^{\gamma}(n+1+k) + \alpha_{(\gamma)}$ for some k.

Choose points in α at $\alpha_{\gamma+1}$, $\alpha_{\gamma+1} + \omega^{\gamma}$, $\alpha_{\gamma+1} + \omega^{\gamma}2, \ldots$, $\alpha_{\gamma+1} + \omega^{\gamma}(m+k)$, and α_{γ}. By induction α_{γ} must map to β_{γ}.

If we map $\alpha_{\gamma+1}$ to a point $\geq \beta_{\gamma+1}$, then some interval of length ω^{γ} corresponds to an interval of length $< \omega^{\gamma}$, contradicting (I) in Proposition 3.2.3.

If we map $\alpha_{\gamma+1}$ to a point $< \beta_{\gamma+1}$ then an interval of length ω^{γ}

corresponds to an interval of length $\geq \omega^{\gamma+1}$, contradicting (I) in Proposition 3.2.3.

(BVc) Suppose $\alpha_{(\gamma)} = \beta_{(\gamma)}$, $m_\gamma \leq n_\gamma$ and $\beta_{\gamma+1} = 0$.
Then $\beta = \omega^\gamma n_\gamma + \beta_{(\gamma)}$. Choose $n_\gamma+1$ initial points of different copies of ω^ρ. Then at least one of the intervals so formed will have to be mapped to an interval of length $< \omega^\gamma$. But, by (I), *not* $\omega^\rho(1+\eta) \leq_{2\gamma+1} \theta$ if θ is of length $< \omega^\gamma$.

(VI) Same as (III). □

The same result is true with $\omega^\rho(1+\eta)$ replaced by $\omega^\rho(1+\omega^*)$ or by $\omega^\rho(1+\omega^*+\omega)$ with virtually the same proof.

Thus we have the following:

Proposition 3.3.6: Let $\alpha > 0$ be a constructive ordinal and let $\mathfrak{A}$ be a recursive linear ordering of type $\omega^\alpha(1+\eta)$, $\omega^\alpha(1+\omega^*)$ or $\omega^\alpha(1+\omega^*+\omega)$. Then if L is the set consisting of the left-most ω^α, L is formally $\Sigma^0_{2\alpha}$ but not formally $\Pi^0_{2\alpha}$.

Proof: Taking a to be the initial element of A, we have from Proposition 3.3.2

$$x \in L \longleftrightarrow \varphi_{2\alpha}(a,x).$$

So L is formally $\Sigma^0_{2\alpha}$. Now consider $R = A - L$.
Given **a**, let a_R be the initial element of the ω^α containing
min $\{ a \in \mathbf{a} : a \in R \}$. Choose a copy of ω^α in R to the left of a_R

and let r be its initial point.

Now given $\mathbf{b}$ and $\beta < \alpha$, let

$$\mathbf{b}_0 = \{\ b \in \mathbf{a},\mathbf{b} : b \in L\ \} \text{ and } b_L = max\ \mathbf{b}_0$$

$$\mathbf{b}_1 = \{\ b \in \mathbf{b} : b \in R\ \&\ b < r\ \}$$

$$\mathbf{b}_2 = \{\ b \in \mathbf{b} : r \le b < a_R\ \} \text{ and } b_R = max\ \mathbf{b}_2$$

$$\text{and } \mathbf{b}_3 = \{\ b \in \mathbf{a},\mathbf{b} : a_R \le b\ \}.$$

In our mapping $\mathbf{a},r,\mathbf{b} \longmapsto \mathbf{a},r',\mathbf{b'}$ we will hold $\mathbf{b}_0$ and $\mathbf{b}_3$ fixed, thus fixing $\mathbf{a}$.

Add to the $\mathbf{b}_1$ the initial element of each copy of ω^α in which some element of $\mathbf{b}_1$ appears, forming $\mathbf{c} = c_1,\dots,\ c_k$ say in sequential order. Map $\mathbf{c}$ into L using the following rule:

$c'_1 = b_L + \omega^\beta$

$c'_{i+1} = c'_i + \theta$ where θ is the interval $[c_i,\ c_{i+1})$ if this is an ordinal less than ω^α and $\theta = \omega^\beta$ otherwise.

Then map r to $r' = c'_k + \omega^\beta$, and for $b \in \mathbf{b}_2$ let $b' = r' + \theta$ where θ is the interval $[r,\ b)$.

Then we may check that $\mathbf{a},r,\mathbf{b} \le_{2\beta+1} \mathbf{a},r',\mathbf{b'}$ as follows.

The interval (b_L,c_1) is $\omega^\alpha(1+\eta)$ [or $\omega^\alpha(1+\omega^*)$], which is $\le_{2\beta+1}$ ω^β = the interval (b_L,c'_1).

Each interval (c_i,c_{i+1}) is either equal to (c'_i,c'_{i+1}) or $\omega^\alpha(1+\eta)$ [or $\omega^\alpha(1+m)$], which is $\le_{2\beta+1}$ ω^β = the interval (c'_i,c'_{i+1}).

The interval (c_k,r) is $\omega^{\alpha}(1+\eta)$ [or $\omega^{\alpha}(1+m)$], which is $\leq_{2\beta+1}$ ω^{β} = the interval (c'_k,r').

The intervals (r,b) and (r',b') are equal for $b \in \mathbf{b}_2$.

And the interval (b_R,a_R) is $\omega^{\alpha}(1+\eta)$ [or $\omega^{\alpha}(1+m)$], which is $\leq_{2\beta+1}$ $\omega^{\alpha}(1+\eta)$ [or $\omega^{\alpha}(1+\omega^*)$] which is (b'_R,a_R).

Since $r' \in L$, this shows that $r \in R - Rcl_{\alpha}(\mathbf{a})$. □

Applying Theorem 2.7.1, we may conclude:

Proposition 3.3.7: Let $\alpha > 0$ be a constructive ordinal. Then there are recursive linear orderings of order type $\omega^{\alpha}(1+\eta)$, $\omega^{\alpha}(1+\omega^*)$ and $\omega^{\alpha}(1+\omega^*+\omega)$ in which the set consisting of the leftmost ω^{α} is not $\Pi^0_{2\alpha}$.

Proof: By Proposition 3.3.5, conditions (1), (2) and (3) of Theorem 2.7.1 are satisfied. By the proof of Proposition 3.3.6, condition (4) is satisfied. □

CHAPTER 4. REDUCED ABELIAN P-GROUPS.

§4.1 Recursive Reduced Abelian P-Groups.

A reduced abelian p-group is an abelian group in which the order of each element is a power of the given prime p and which contains no divisible subgroup. That is, there is no infinite sequence of distinct elements $y_1,y_2,\ldots$ such that $py_{i+1}=y_i$. For a recursive reduced abelian p-group, the group operation (addition) must be a recursive function on a recursive set of group elements. This ensures that inverses are recursive, as $-x = (p^n-1)x$ if $p^n x = 0$.

Note that as we are dealing with operations rather than relations here, a group is recursive iff it is r.e. Consequently we consider recursive groups.

Our basic reference for infinite abelian group theory is Kaplansky [Ka].

For a reduced abelian p-group G, define G_α as follows:

$$G_0 = G,\ G_{\alpha+1} = pG_\alpha, \text{ and for limit ordinals } \alpha,\ G_\alpha = \bigcap_{\beta<\alpha} G_\beta.$$

Let $P = \{\ x \in G : px=0\ \}$, and for any subgroup $S \leq G$, let

$$S_\alpha = S \cap G_\alpha \text{ and } S^*_\alpha = S_\alpha \cap p^{-1}G_{\alpha+2}.$$

The *height* of an element $x \neq 0$, $h(x)$, is defined to be the unique α for which $x \in G_\alpha$ and $x \notin G_{\alpha+1}$. Conventionally $h(0) = \infty$, where ∞ is greater than any α. The *length* of G, $\lambda(G)$, is the smallest α for which $G_\alpha = \{0\}$.

We have the following inequalities concerning height:

(a) If $h(x) < h(y)$, then $h(x+y) = h(x)$.

(b) If $h(x) = h(y)$, then $h(x+y) \geq h(x)$.

(c) If $p \nmid r$, then $h(rx) = h(x)$.

(d) If $x \neq 0$, then $h(px) > h(x)$.

If S is any subgroup of G and $x \in G$, we say x is *proper* with respect to S if $h(x) \geq h(x+s)$ for every $s \in S$; that is, if x has maximal height in its coset mod S. This implies that in these circumstances $h(x+s)$ is actually equal to $min\ \{h(x),\ h(s)\}$.

The Ulm invariants of G are defined as

$$U_\alpha(G) = dim\ (P_\alpha/P_{\alpha+1})$$

where the quotient group $P_\alpha/P_{\alpha+1}$ is regarded as a vector space over $\mathbb{Z}_p$.

A celebrated result in abelian group theory is

Theorem 4.1: (Ulm's Theorem) Two countable reduced abelian p-groups are isomorphic iff they have the same Ulm invariants.

Proof: See [Ka]. ■

§4.2 Back and Forth Relations.

The result of this section is in some sense a generalization of Ulm's Theorem, and uses a construction similar to but more complicated than that of the proof given in Kaplansky [Ka]. It bears some similarity to the result in [BwE], but differs in that there formulae are ranked by quantifier rank rather than by position in our hyperarithmetical hierarchy, and the interest is in when two groups satisfy the same formulae of a given quantifier rank. By contrast, our back and forth relations are not symmetric.

We require the following result from [Ka] (Lemma 13). Kaplansky observes that if S is any subgroup of G and $a \in S^*_\alpha$, then there is $b \in G_{\alpha+1}$ such that $pa = pb$. The map $a \mapsto a-b$ followed by the natural homomorphism from P_α onto $P_\alpha/P_{\alpha+1}$ is a homomorphism of S^*_α into $P_\alpha/P_{\alpha+1}$ whose kernel is exactly $S_{\alpha+1}$. We thus have an isomorphism, U say, of $S^*_\alpha/S_{\alpha+1}$ into $P_\alpha/P_{\alpha+1}$.

Lemma 4.2.1: The following two statements are equivalent:

(a) The range of U is not all of $P_\alpha/P_{\alpha+1}$.

(b) There exists in P_α an element of height α proper with respect to S.

Proof: See Kaplansky [Ka]. □

The back and forth relations apply to finite sequences of the same length. For such sequences **a** and **b**, let f : **b** $\longrightarrow$ **a** map corresponding members of the sequence onto each other.

Proposition 4.2.2:

(i) **a** $\leq_{2\alpha}$ **b** iff f extends to an isomorphism $f : \langle\mathbf{b}\rangle \cong \langle\mathbf{a}\rangle$ and for every $b \in \langle\mathbf{b}\rangle$ and $a = f(b)$ we have

$$h(b) = h(a) < \omega\alpha \quad or \quad h(b),h(a) \geq \omega\alpha.$$

(ii) **a** $\leq_{2\alpha+1}$ **b** iff f extends to an isomorphism $f : \langle\mathbf{b}\rangle \cong \langle\mathbf{a}\rangle$ and for every $b \in \langle\mathbf{b}\rangle$ and $a = f(b)$ we have

(iia) In the case that $P_{\omega\alpha+k}$ is infinite for every $k<\omega$

$$h(b) = h(a) < \omega\alpha, \text{ or}$$

$$h(b) \geq \omega\alpha \text{ and } h(a) \geq min\ \{h(b),\ \omega\alpha+\omega\}.$$

(iib) In the case that $P_{\omega\alpha+k}$ is infinite and $P_{\omega\alpha+k+1}$ is finite

$$h(b) = h(a) < \omega\alpha, \text{ or}$$

$$\omega\alpha \leq h(b) \leq h(a) \leq k, \text{ or}$$

$$h(b)=h(a) > \omega\alpha+k.$$

(iic) In the case that $P_{\omega\alpha}$ is finite

$$h(b) = h(a).$$

Proof: The proof is by induction on α, and in two parts.

Part A: We show that if **b**,**a** satisfy the above conditions for $\gamma+1$, then given any **d** we can find **c** so that **b**,**d** $\leq_{\gamma}$ **a**,**c**. [For limit ordinals α, given $\beta < \alpha$ and **d**, we can find **c** so that **b**,**d** $\leq_{\beta}$ **a**,**c**.]

Part B: Conversely, we show that if **b**,**a** do not satisfy the above conditions for $\gamma+1$, we can find **d** so that *not* **b**,**d** $\leq_{\gamma}$ **a**,**c** for any choice of **c**. [For limit ordinals α, there is some $\beta < \alpha$ such that *not* **a** $\leq_{\beta}$ **b**.]

A. The situation is that we have finite subgroups $B = \langle \mathbf{b} \rangle$, $A = \langle \mathbf{a} \rangle$ of G, an isomorphism $f : B \cong A$ satisfying a given condition on the heights of elements and their images, and a finite extension $B' = \langle \mathbf{b},\mathbf{d} \rangle \geq B$. We need to extend f to B' so that it now satisfies a different condition. Except in the case of α a limit ordinal, which is easier, we will do this in stages, forming a chain of groups $B = B^0 < B^1 < \ldots < B^n = B'$, by choosing an appropriate $x \in B' - B^i$ with $px \in B^i$ and extending f to $B^{i+1} = \langle x, B^i \rangle$.

(Aia) Suppose α is a limit ordinal. Let B,A be finite subgroups of G and let $f : B \cong A$ satisfy

$$h(b) = h(a) < \omega\alpha \text{ or } h(b),h(a) \geq \omega\alpha \text{ for all } b \in B,\ a = f(b).$$

If $h(b) = h(a)$ for all $b \in B$, then f is height preserving, and so can be extended to an automorphism of G ([Ka], problem 38). So certainly, given $\mathbf{d}$ we can find $\mathbf{c}$ so that $\mathbf{b},\mathbf{d} \leq_\beta \mathbf{a},\mathbf{c}$ for any β.

If $h(b) \geq \omega\alpha$ for some $b \neq 0$, then $P_{\omega\alpha}$ is non-null, so $P_{\omega\beta}$ is infinite for every $\beta < \alpha$ ([Ka], problem 36). Given $\beta < \alpha$ and $\mathbf{d}$, we see that f satisfies (i) or (iia) for β+1. So by induction $\mathbf{a} \leq_{\beta+1} \mathbf{b}$, so there is $\mathbf{c}$ such that $\mathbf{b},\mathbf{d} \leq_\beta \mathbf{a},\mathbf{c}$.

(Aib) Suppose $\alpha = \beta+1$. let $B = \langle \mathbf{b} \rangle$, $A = \langle \mathbf{a} \rangle$ be finite subgroups of G and let $f : B \cong A$ satisfy

$$h(b) = h(a) < \omega\alpha \text{ or } h(b),h(a) \geq \omega\alpha \text{ for every } b \in B,\ a = f(b).$$

If $h(b) = h(a)$ for every $b \in B$, then f can be extended to an automorphism of G, so given **d** we can find **c** so that $\mathbf{b},\mathbf{d} \leq_{2\beta+1} \mathbf{a},\mathbf{c}$.

If $h(b) \geq \omega\alpha$ for some b, then $P_{\omega\alpha}$ is non-null, so $P_{\omega\beta+k}$ is infinite for every $k < \omega$. Given $B' \geq B$, let

$$M = max\{m{:}b'{\in}B',\ h(b'){=}\omega\beta{+}m \text{ and } m{<}\omega\} + 1,\ N_0{=}Card(B'-B){+}M$$
$$\text{and } N_i{=}N_0{-}i.$$

We see that $f{:}B{\cong}A$ satisfies

$$(*\mathrm{i}) \qquad h(b){=}h(a) \leq \omega\beta{+}M, \text{ or } h(b) \geq \omega\alpha \text{ and } h(a) \geq \omega\beta{+}N_i$$
$$\text{for every } b{\in}B^i,\ a{=}f(b)$$

with $i{=}0$. We will ensure that each extension of f to B^i satisfies (*i). Then $f{:}B'{\cong}A'$ will satisfy

$$h(b'){=}h(a') < \omega\beta, \text{ or } h(a') \geq \omega\beta \text{ and } h(b') \geq min\{h(a'),\ \omega\beta{+}\omega\}$$
$$\text{for every } b'{\in}B',\ a'{=}f(b').$$

That is, if $\mathbf{c}{=}f(\mathbf{d})$, $\mathbf{b},\mathbf{d} \leq_{2\beta+1} \mathbf{a},\mathbf{c}$ as required.

Here is the construction. Take $x{\in}B'{-}B^i$ such that $px{\in}B^i$, x is proper with respect to B^i and $h(px)$ is maximal amongst all elements in $x{+}B^i$ for which this is true. (Such an x exists because $B'{-}B^i$ is finite and the coset $x{+}B^i$ is finite.) Let $f(px){=}y$.

Now either $h(px)=h(y) \le \omega\beta+M$, or $h(px) \ge \omega\alpha$ and $h(y) \ge \omega\beta+N_i$.

Case 1. $h(y)=h(px)=h(x)+1 \le \omega\beta+M$.

We have $y\neq 0$, $px\neq 0$. Choose w so that $h(w)=h(x)$ and $pw=y$.

Now $w\notin A^i$, for if $w=f(z)$ with $z\in B^i$, then $px=pz$ as they both map to y; also $x-z\notin B^i$ for otherwise x would be in B^i; and $h(x-z)=h(x)$ because $h(z)=h(w)=h(x)$ and x is proper with respect to B^i. But $h(px-pz)=h(0) > h(px)$, which contradicts the maximal choice of $h(px)$.

Also w is proper with respect to A^i. Suppose, on the contrary, that $h(w+a) \ge h(x)+1$, where $a\in A^i$, $a=f(b)$ for $b\in B^i$. Since $w+a\neq 0$, we have $h(pw+pa) \ge h(x)+2$, hence $h(px+pb) \ge h(x)+2$. But this contradicts the maximal height of px.

Now extend f to $B^{i+1}=\langle x,B^i\rangle$ as follows

$$f(rx+b) = rw+a \quad \text{for } 0<r<p,\ b\in B^i \text{ and } a=f(b).$$

Since w is proper with respect to A^i, we see that f still preserves heights below $\omega\beta+M$, thus f satisfies (*i+1).

Case 2. $h(x) \ge \omega\alpha$ and $h(y) \ge \omega\beta+N_i$.

Choose w_1 with $pw_1=y$ and $h(w_1) \ge \omega\beta+N_i-1 = \omega\beta+N_{i+1}$. Since $P_{\omega\alpha+N_i}$ is infinite, there is $w_2\in P_{\omega\alpha+N_i} - A^i$. Put $w=w_1+w_2$. Then $w \notin A^i$, $pw = y$ and $h(w) \ge \omega\beta+N_i-1$, so extend f by $f(x)=w$.

Claim: f satisfies (*i+1).

[Check: If $h(b) \le \omega\beta+M$ then $h(a)=h(b)$, so $h(rx+b)=h(b)$ and

$h(rw+a)=h(a)$.

If $h(b) \geq \omega\alpha$ then $h(a) \geq \omega\beta+N_i > \omega\beta+N_{i+1}$, so $h(rx+b) \geq \omega\alpha$ and $h(rw+a) \geq \omega\beta+N_{i+1}$.

Note that M is chosen so that for no b do we have $\omega\beta+M<h(b)<\omega\alpha$.]

Case 3. $\gamma=h(x) < \omega\alpha$ and $h(px) > \gamma+1$.

Since $h(px) > \gamma+1$, there is $v \in G_{\gamma+1}$ such that $pv=px$. The element $x-v$ is in P_γ; like x it has height γ; and also it is proper with respect to B^i (since v does not interfere in computations of height $\leq \gamma$). We now apply Lemma 4.2.1. Since $B^{i*}_\gamma/B^i_{\gamma+1}$ is finite, its dimension as a vector space over $\mathbb{Z}_p$ is strictly less than the γth Ulm invariant of G, $U_\gamma(G)$.

Since f preserves heights $\leq \omega\beta+M$ and $\gamma\leq\omega\beta+M-1$, f maps B^i_γ onto A^i_γ, B^{i*}_γ onto A^{i*}_γ and $B^i_{\gamma+1}$ onto $A^i_{\gamma+1}$. Thus the dimension of $A^{i*}_\gamma/A^i_{\gamma+1}$ is also less than $U_\gamma(G)$. Applying Lemma 4.2.1 again, we have that G contains an element w_1 for which $pw_1=0$, $h(w_1)=\gamma$ and which is proper with respect to A^i.

Now either $h(y)=h(px) > \gamma+1$ or $h(y) \geq \omega\beta+N_i > \omega\beta+M > h(x)+1=\gamma+1$.

So $y=pw_2$ with $w_2 \in G_{\gamma+1}$. Take $w=w_1+w_2$. Then $pw=pw_1+pw_2=0+y=y$, $h(w)=h(w_1)=\gamma$, and w is proper with respect to A^i.

Extending f by $f(x)=w$, we see that f still preserves heights $\leq \omega\beta+M$, and so satisfies (*i+1).

(Aiia) Suppose $P_{\omega\alpha+k}$ is infinite for every $k<\omega$.

Let $B=\langle b\rangle$, $A=\langle a\rangle$ be finite subgroups of G and let $f:B\cong A$ satisfy

$$h(b)=h(a) < \omega\alpha, \text{ or } h(b) \geq \omega\alpha \text{ and } h(a) \geq min\{h(b), \omega\alpha+\omega\}$$

$$\text{for every } b \in B \text{ and } a=f(b).$$

Given a finite extension $B' \geq B$, let $N_0 = Card(B'-B)$ and $N_i = N_0 - i$. We see that f satisfies

(*i) $\quad h(b)=h(a) < \omega\alpha$, or $h(b) \geq \omega\alpha$ and $h(a) \geq min\{h(b), \omega\alpha+N_i\}$

with $i=0$. We will ensure that each extension of f to B^i satisfies (*i). Then $f:B' \cong A'$ will satisfy

$$h(b')=h(a') < \omega\alpha, \text{ or } h(b'),h(a') \geq \omega\alpha \text{ for every } b' \in B'.$$

That is, if $\mathbf{c}=f(\mathbf{d})$, $\mathbf{b},\mathbf{d} \leq_{2\alpha} \mathbf{a},\mathbf{c}$ as required.

Here is the construction. Again take $x \in B'-B^i$ such that $px \in B^i$, x is proper with respect to B^i and $h(px)$ is maximal amongst all elements in $x+B^i$ for which this is true. (Such an x exists because $B'-B^i$ is finite and the coset $x+B^i$ is finite.) Let $f(px)=y$.
Now either $h(px)=h(y) < \omega\alpha$, or $h(px) \geq \omega\alpha$ and $h(y) \geq min\{h(px), \omega\alpha+N_i\}$.

Case 1. $h(y)=h(px)=h(x)+1 < \omega\alpha$.
Choose w so that $pw=y$ and $h(w)=h(x)$. Then as before $w \notin A^i$ and w is proper with respect to A^i, so extend f by $f(x)=w$. Then again f preserves heights $< \omega\alpha$, and so satisfies (*i+1).

Case 2. $h(x) < \omega\alpha$ and $h(px) > h(x)+1$.

Since f preserves heights $< \omega\alpha$, the argument of (Aib) Case 3 applies.

Case 3. $h(x) \geq \omega\alpha$ and $h(y) \geq min\{h(px), \omega\alpha+N_i\}$.

Now $h(px) \geq h(x)+1$, so there is w_1 with $pw_1=y$ and $h(w_1) \geq min\{h(x), \omega\alpha+N_i-1\}$. Since $P_{\omega\alpha+N_i}$ is infinite, there is $w_2 \in P_{\omega\alpha+N_i}-A^i$. Put $w=w_1+w_2$, and extend f by $f(x)=w$.

Claim: f satisfies (*i+1).

[Check: Clearly f still preserves heights $< \omega\alpha$.

Also if $h(a) \geq min\{h(b), \omega\alpha+N_i\}$ then

$$h(rw+a) \geq min\{h(w), h(a)\} \geq min\{h(x), \omega\alpha+N_i-1, h(b), \omega\alpha+N_i\}$$
$$\geq min\{h(x), h(b), \omega\alpha+N_{i+1}\} = min\{h(rx+b), \omega\alpha+N_{i+1}\}$$

since x is proper with respect to B^i.]

(Aiib). Suppose $P_{\omega\alpha+k}$ is infinite and $P_{\omega\alpha+k+1}$ is finite.

Let $B=\langle \mathbf{b} \rangle$, $A=\langle \mathbf{a} \rangle$ be finite subgroups of G and let $f:B\cong A$ satisfy

(*) $h(b)=h(a) < \omega\alpha$, or $\omega\alpha \leq h(b) \leq h(a) \leq \omega\alpha+k$, or $h(b)=h(a) > \omega\alpha+k$ for every $b\in B$ and $a=f(b)$.

Given a finite extension $B' \geq B$, we will ensure that each extension of f to B^i satisfies (*). Then $f:B'\cong A'$ will satisfy

$$h(b')=h(a') < \omega\alpha \text{ or } h(b'), h(a') \geq \omega\alpha \text{ for every } b'\in B'.$$

That is, if $\mathbf{c}=f(\mathbf{d})$, $\mathbf{b},\mathbf{d} \leq_{2\alpha} \mathbf{a},\mathbf{c}$ as required.

Here is the construction. Again take $x \in B'-B^1$ such that $px \in B^1$, x is proper with respect to B^1 and $h(px)$ is maximal amongst all elements in $x+B^1$ for which this is true. (Such an x exists because $B'-B^1$ is finite and the coset $x+B^1$ is finite.) Let $f(px)=y$. Now either $h(px)=h(y) < \omega\alpha$, or $\omega\alpha \leq h(px) \leq h(y) \leq \omega\alpha+k$, or $h(px)=h(y) > \omega\alpha+k$.

Case 1. $h(y)=h(px)=h(x)+1 < \omega\alpha$.

Case 2. $h(x) < \omega\alpha$ and $h(px) > h(x)+1$.

Both are dealt with in the same way as in (Aiia).

Case 3. $\omega\alpha \leq h(x) \leq \omega\alpha+k$ and $\omega\alpha \leq h(px) \leq h(y)$.

Choose w_1 with $pw_1=y$ and $h(w_1) \geq h(x)$.

Now $P_{\omega\alpha+k}$ is infinite and $P_{\omega\alpha+k+1}$ is finite, so the dimension of $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$ is infinite. Let $C=\langle w_1, A^1\rangle$. Since $C^*_{\omega\alpha+k}/C_{\omega\alpha+k+1}$ is finite, we can apply Lemma 4.2.1 to find an element $w_2 \in P_{\omega\alpha+k}$ of height $\omega\alpha+k$ proper with respect to C.

Extend f by $f(x)=w=w_1+w_2$. We claim that f still satisfies (*).

[Check: Clearly f still preserves heights below $\omega\alpha$.

Now $h(rw+a) = min\{h(w_2), h(rw_1+a)\} \leq \omega\alpha+k$

and $h(rw+a) \geq min\{h(w_1), h(w_2), h(a)\} \geq min\{h(x), h(b)\} = h(rx+b)$

since $h(w_2)$, $h(w_1) \geq h(x)$; $h(a) \geq h(b)$, and x is proper with respect to B^1.]

Case 4. $h(x) > \omega\alpha+k$ and $h(y)=h(px)=h(x)+1$.

Case 5. $h(x) > \omega\alpha+k$ and $h(y)=h(px) > h(x)+1$.

Since f preserves heights above $\omega\alpha+k$, these are both handled as cases 1 and 2 above.

(Aiic) If f is height preserving then $\mathbf{a} \leq_\beta \mathbf{b}$ for any β.

B. If the function f:**b**⟶**a** does not extend to an isomorphism, then there is a quantifier free formula satisfied by **b** and not **a**, so certainly not $\mathbf{a} \leq_\gamma \mathbf{b}$ for any γ. So assume that f extends to an isomorphism, but fails to satisfy the other conditions.

(Bia). If α is a limit ordinal and f does not satisfy the condition, then there are $b \in B$, $a=f(b)$ such that $h(b) \neq h(a)$ and $h(a)<\omega\alpha$ or $h(b)<\omega\alpha$. But then $h(a)<\omega\beta$ or $h(b)<\omega\beta$ for some $\beta<\alpha$. Hence, by induction, not $\mathbf{a} \leq_{2\beta} \mathbf{b}$.

(Bib). If $\alpha=\beta+1$ and f does not satisfy the condition then there are $b \in B$, $a=f(b)$ such that $h(a)<\omega\alpha$ and $h(a)<h(b)$, or $h(b)<\omega\alpha$ and $h(b)<h(a)$.

Case 1a. $h(a)<\omega\beta$ and $h(a)<h(b)$.

Not $\mathbf{a} \leq_{2\beta} \mathbf{b}$, by induction.

Case 1b. $h(a)=\omega\beta+k$ for some $k<\omega$ and $h(a)<h(b)$.

We can find $x \in G_{\omega\beta}$ such that $p^{k+1}x=b$, but cannot find any $y \in G_{\omega\beta}$ with $p^{k+1}y=a$. So not $\mathbf{b},x \leq_{2\beta+1} \mathbf{a},y$ for any y.

Case 2a. $h(b)<\omega\beta$ and $h(b)<h(a)$.

Then not $\mathbf{b} \le_{2\beta} \mathbf{a}$ by induction, so not $\mathbf{a} \le_{2\beta+1} \mathbf{b}$.

Case 2b. $h(b)=\omega\beta+k$ for some $k<\omega$ and $h(b)<h(a)$.

We can find $x\in G_{\omega\beta}$ such that $p^{k+1}x=a$, but cannot find any $y\in G_{\omega\beta}$ with $p^{k+1}y=b$. So not $\mathbf{a},x \le_{2\beta} \mathbf{b},y$ for any y. Hence not $\mathbf{b} \le_{2\beta+1} \mathbf{a}$, so not $\mathbf{a} \le_{2\alpha} \mathbf{b}$.

(Biia). If $P_{\omega\alpha+k}$ is infinite for every $k<\omega$, and f does not satisfy the condition, then there are $b\in B$, $a=f(b)$ such that

1. $h(a)<\omega\alpha$, $h(a)<h(b)$ or
2. $h(b)<\omega\alpha$, $h(b)<h(a)$ or
3. $h(a)=\omega\alpha+m < h(b)$ with $m<\omega$.

Cases 1,2. Here not $\mathbf{a} \le_{2\alpha} \mathbf{b}$, so not $\mathbf{a} \le_{2\alpha+1} \mathbf{b}$.

Case 3. We can find $x\in G_{\omega\alpha}$ such that $p^{m+1}x=b$, but can not find any $y\in G_{\omega\alpha}$ with $p^{m+1}y=a$. So not $\mathbf{b},x \le_{2\alpha} \mathbf{a},y$ for any y.

(Biib). If $P_{\omega\alpha+k}$ is infinite and $P_{\omega\alpha+k+1}$ is finite, and f does not satisfy the condition, then there are $b\in B$, $a=f(b)$ such that

1. $h(a)<\omega\alpha$, $h(a)<h(b)$ or
2. $h(b)<\omega\alpha$, $h(b)<h(a)$ or
3. $h(a)=\omega\alpha+m < h(b)$ with $m<\omega$ or
4. $h(a)=\omega\alpha+m > h(b)$ with $k<m<\omega$.

Cases 1,2,3. As for (Biia).

Case 4. We have $P_{\omega\alpha+m}$ is finite, so $G_{\omega\alpha+m}$ is finite. If **d** is chosen so that **b**,**d** includes all elements of $G_{\omega\alpha+m}$, then any extension of f to <**b**,**d**> must map an element of $G_{\omega\alpha+m}$, say g, to an element of lesser height, say h. We can find $x \in G_{\omega\alpha}$ such that $p^m x = g$, but can not find $y \in G_{\omega\alpha}$ with $p^m y = h$.

So not **b**,**d**,$x \leq_{2\alpha}$ **a**,**c**,y for any y.

(Biic). If $P_{\omega\alpha}$ is finite and f does not satisfy the condition, then there are $b \in B$, $a = f(b)$ such that

1. $h(a) < \omega\alpha$, $h(a) < h(b)$ or
2. $h(b) < \omega\alpha$, $h(b) < h(a)$ or
3. $h(a) = \omega\alpha + m < h(b)$ with $m < \omega$ or
4. $h(a) = \omega\alpha + m > h(b)$ with $m < \omega$.

These are all dealt with as the corresponding cases of (Biib). ■

§4.3 Recursive Height Functions.

To apply Proposition 4.2.2 to Theorem 2.7.1 to obtain examples of relations on a reduced abelian p-group G that are or are not intrinsically Σ^0_α, we need the relations $\leq_\gamma$ to be uniformly r.e. for $\gamma < \alpha$. To achieve this it suffices that the height function $h\colon G \longrightarrow \omega_1^{CK}$ be recursive. This will also ensure that the existential diagram of G is recursive.

The following results from [AKnO], which basically restate work from [Rol], show that for any reduced abelian p-group with a recursive sequence of Ulm invariants we can construct a recursive copy for which the height function is recursive.

Let $T \subseteq \omega^{<\omega}$ be a tree of finite sequences from ω, ordered by inclusion, with no infinite branches.

Let $G(T)$ be the abelian group freely generated by the nodes of T under the relations:

i) $\phi = 0$, the group identity, where ϕ is the root node of the tree.

ii) $pa = b$ where a is a successor of b.

Then $G(T)$ is a reduced abelian p-group.

Define the *height* of a node in T as follows:

i) a has height 0 if a is terminal.

ii) a has height $\alpha+1$ if it has a successor of height α and all its successors have height $\leq \alpha$.

iii) a has height α, for limit ordinals α, if for arbitrarily large $\beta < \alpha$, a has a successor of height β, and all its successors have height $< \alpha$.

The *height* of T is the height of the root node, ϕ, in T.

Proposition 4.3.1: For any $T \subseteq \omega^{<\omega}$ with no infinite branches, if $G = G(T)$ then $\lambda(G)$ = height of T, and for any $\beta < \lambda(G)$, $U_\beta(G)$ is the number of nodes $a \in T$ such that a has height β and one of the following holds:

(a) a is at level 1 in t;

(b) a is a successor of some node of height $> \beta+1$:

(c) a is a successor of some node b of height $\beta+1$ and a is not the first successor of b of height β. □

Proposition 4.3.2: For any countable reduced abelian p-group G, $G = G(T)$ for some $T \in \omega^{<\omega}$ with no infinite branches.

The proof of Proposition 4.3.2 will show us how to construct the recursive group we require.

Proof: [Oates] First we note that G is determined by its Ulm invariants. Furthermore if $\alpha < \beta \leq \lambda(G)$ are limit ordinals then the Ulm invariants of G must be non-zero for infinitely many ordinals between α and β. ([Ka], Exercise 36.)

Fix a list of all the pairs (β,k) such that $k \leq U_\beta(G)$, $1 \leq k < \omega$ and so that (β,k) precedes $(\beta,k+1)$. We shall construct $T \subseteq \omega^{<\omega}$ so that $G = G(T)$. We shall label the nodes of T to indicate height and also put * beside certain nodes. (If a node has label $\beta+1$ then one of its successors will receive * as well as the label β.)

We say that a pair (β,k) is *taken care of* if there are k nodes a with label β and situated either (i) at level 1, or (ii) the successor of some node with label $\gamma > \beta+1$, or (iii) the successor of some node with label $\beta+1$ and a does not have *.

Note that we can take care of (β,k) by adding just one new node, after we have taken care of (β,j) for $j < k$.

A node a is said to *need attention* if one of the following holds:

(a) a has label $\beta+1$ and there is no successor node with label β and *. (If we provide such a node, we are not taking care of any new pair on the list.)

(b) a has label β, where β is a limit ordinal. These nodes always need attention. We keep adding successors b_n labelled with new ordinals $\gamma_n < \beta$ such that the pair $(\gamma_n,1)$ is not yet taken care of and $\lim \gamma_n = \beta$. The constraint on the sequence of Ulm invariants noted above guarantees that there will always be such a pair $(\gamma_n,1)$.

We begin with the root node without label, to represent the group identity. The construction proceeds in stages. At each stage, we take the first pair on the list that has not been taken care of already, and add an appropriate node to the tree. (It does not matter how this is done, provided each node succeeds one of strictly greater label. For definiteness, we could add each node in this part of the construction as a successor to the root node.) Then for each node currently in the tree which needs attention we add one node. If this is under case (b) above, we note the pair $(\gamma_n,1)$ which is thus taken care of.

Let T be the tree resulting from this construction. Then it is clear that:

(1) For each node with label $\beta+1$, there is a successor with label β and *.

(2) For each node with label β, for β a limit ordinal, and each $\delta < \beta$, there is a successor with label γ for some γ between δ and β.

(3) Every node in T without * was added to take care of some pair, and every node with * was added to attend to the node above (witnessing height).

From this and Proposition 4.3.1 it follows that $G(T) = G$. □

Proposition 4.3.3: If G is a countable reduced abelian p-group with a recursive sequence of Ulm invariants, then there is a recursive copy of G for which the height function is recursive.

Proof: If the list of pairs (β,k) in the above proof is recursive then the whole construction is effective. But this is clearly possible when the Ulm invariants are recursive.

Furthermore we may extend the construction of the tree to construct the associated group at the same time. The group consists of all finite combinations of tree elements

$$\sum_{j<n} r_j a_j \quad \text{where } 0 \le r_j < p \text{ and } a_j \in T.$$

Addition is defined by the rule

$$r_j a_j + s_j a_j = \begin{cases} (r_j+s_j)a_j & \text{if } r_j+s_j < p \\ (r_j+s_j-p)a_j + A_j & \text{if } r_j+s_j \ge p \end{cases}$$

where A_j is the predecessor of a_j, together with associativity and commutativity. Also the height of $\sum_{j<n} r_j a_j$ is just

$$min\ \{h(a_j) : r_j \neq 0\ \}.$$

So each time an element is added to the tree, we add all such combinations of it and all the (finitely many) elements already in the group, noting the height, to the construction of the group, building up the group addition table as we go. □

CHAPTER 5. EXAMPLES IN REDUCED ABELIAN P-GROUPS.

§5.1 Intrinsically Σ^0_α Relations.

As an application of Theorem 2.7.1 we will examine various sets of elements of reduced abelian p-groups.

We will consider countable reduced abelian p-groups with recursive Ulm invariants. For such a group, by Proposition 4.3.3, there is a recursive copy of the group for which conditions (1), (2) and (3) of Theorem 2.7.1 are satisfied. If the relation under consideration is not formally Σ^0_α, we will show this by showing that condition (4) of Theorem 2.7.1 is satisfied.

Proposition 5.1.1: In any recursive reduced abelian p-group G

The set of elements of height $\geq \omega\alpha$, $G_{\omega\alpha}$, is formally $\Pi^0_{2\alpha}$.

The set of elements of height $\geq \omega\alpha+n$, $G_{\omega\alpha+n}$, is formally $\Sigma^0_{2\alpha+1}$.

The set of elements of order p of height $\geq \omega\alpha$, $P_{\omega\alpha}$, is formally $\Pi^0_{2\alpha}$.

The set of elements of order p of height $\geq \omega\alpha+n$, $P_{\omega\alpha+n}$, is formally $\Sigma^0_{2\alpha+1}$.

Proof: We have the following formulae:

$$x \in G_n \underset{df}{\longleftrightarrow} \exists y\ (\ p^n y = x\ )$$

$$x \in G_\omega \underset{df}{\longleftrightarrow} \bigwedge_{n<\omega} \exists y\ (\ p^n y = x\ )$$

$$x \in G_{\omega\alpha+n} \underset{df}{\longleftrightarrow} \exists y\ (\ p^n y = x\ \&\ y \in G_{\omega\alpha}\ )$$

$$x \in G_{\omega\alpha+\omega} \underset{df}{\longleftrightarrow} \bigwedge_{n<\omega} \exists y\ (\ p^n y = x\ \&\ y \in G_{\omega\alpha}\ )$$

$$x \in G_{\omega\alpha} \underset{df}{\longleftrightarrow} \bigwedge_{\gamma<\alpha} (\ x \in G_{\omega\gamma}\ ) \qquad \text{for limit ordinals } \alpha$$

Thus by induction we see that $G_{\omega\alpha}$ is defined by a recursive $\Pi_{2\alpha}$ formula, and $G_{\omega\alpha+n}$ is defined by a recursive $\Sigma_{2\alpha+1}$ formula.
Furthermore $x \in P_\beta \longleftrightarrow x \in G_\beta$ & $px = 0$. □

Proposition 5.1.2: If $G_{\omega\alpha}$ is infinite then $G_{\omega\alpha}$ and $P_{\omega\alpha}$ are not formally $\Sigma^0_{2\alpha}$.

Proof: We can do both cases simultaneously. Since $G_{\omega\alpha}$ is infinite so is $P_{\omega\alpha}$. Let $R = G_{\omega\alpha}$ (or $P_{\omega\alpha}$).
Given **a**, we want to find $r \in R - Rcl_{2\alpha}(\mathbf{a})$.

Case (i): $\alpha = \beta+1$.

Given **a**, choose $r \in P_{\omega\alpha} - \langle\mathbf{a}\rangle$. We show that r is the element we seek.
Given **c**, consider $C = \langle\mathbf{a},r,\mathbf{c}\rangle \geq \langle\mathbf{a}\rangle = A$. Let

$$M = max\ \{\ m : c\in C,\ h(c)=\omega\beta+m \text{ and } m<\omega\ \} +1,\ N_0 = Card(C-A) +M$$

Choose $r' \in P_{\omega\beta+N_0} - P_{\omega\alpha} - \langle\mathbf{a}\rangle$, proper with respect to $\langle\mathbf{a}\rangle$.
Such an r' exists because $U_{\omega\beta+N}(G)$ is nonzero for infinitely many $N < \omega$.
We want to find **c'** so that $\mathbf{a},r,\mathbf{c} \leq_{2\beta+1} \mathbf{a},r',\mathbf{c}'$.
This amounts to extending the identity function $f\colon A \cong A$ to an isomorphism with domain C satisfying $f(r) = r'$ and

$$h(c) = h(s) < \omega\beta \text{ or}$$

$$h(s) \geq \omega\beta \text{ and } h(c) \geq min \{ h(s), \omega\alpha \}$$

$$\text{for every } c \in C,\ s = f(c).$$

This is done in stages exactly as the construction in (Aib) of Proposition 4.2.2.

Case (ii): α is a limit ordinal.

Given $\mathbf{a}$, choose $r \in P_{\omega\alpha} - \langle \mathbf{a} \rangle$.

Given $\beta < \alpha$, choose $r' \in P_{\omega\beta+\omega} - P_{\omega\alpha} - \langle \mathbf{a} \rangle$.

Then $\mathbf{a},r' \leq_{2\beta+2} \mathbf{a},r$, so given $\mathbf{c}$ there is $\mathbf{c}'$ such that

$$\mathbf{a},r,\mathbf{c} \leq_{2\beta+1} \mathbf{a},r',\mathbf{c}'.$$

In both cases we have demonstrated that the chosen r is an element of $R - Rcl_{\alpha}(\mathbf{a})$. The existence of such an r for any $\mathbf{a}$ shows that $G_{\omega\alpha}$ (respectively $P_{\omega\alpha}$) is not formally $\Sigma^0_{2\alpha}$. □

Thus we may apply Theorem 2.7.1 and conclude that

Proposition 5.1.3: If G is a recursive reduced abelian p-group with recursive height function in which $G_{\omega\alpha}$ (or $P_{\omega\alpha}$) is infinite, then there is an isomorphic recursive reduced abelian p-group H for which $H_{\omega\alpha}$ (respectively $H_{\omega\alpha} \cap \{h : ph = 0\}$) is not a $\Sigma^0_{2\alpha}$ set. □

We have similar but more complicated results for $G_{\omega\alpha+n}$ and $P_{\omega\alpha+n}$, to which we now turn.

Proposition 5.1.4: Suppose $P_{\omega\alpha+n}$ is infinite.

(i) If $U_{\omega\alpha+k}(G) = 0$ for every $0 \leq k < n$, then $P_{\omega\alpha+n}$ is formally $\Pi^0_{2\alpha}$, but not formally $\Sigma^0_{2\alpha}$.

(ii) If $U_{\omega\alpha+k}(G) \neq 0$ for some $0 \leq k < n$, and $U_{\omega\alpha+k}(G) < \aleph_0$ for each $0 \leq k < n$, then $P_{\omega\alpha+n}$ is formally $\Delta^0_{2\alpha+1}$, but not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$.

(iii) If $U_{\omega\alpha+k}(G) = \aleph_0$ for some $k < n$, then $P_{\omega\alpha+n}$ is formally $\Sigma^0_{2\alpha+1}$ but not formally $\Pi^0_{2\alpha+1}$.

Proof: (i) If $U_{\omega\alpha+k}(G) = 0$ for every $0 \leq k < n$, then $P_{\omega\alpha+n} = P_{\omega\alpha}$.

(ii) Let $\mathbf{p}^k$ be sets of representatives of bases for $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$. Then we claim that we have the following $\Pi_{2\alpha+1}$ formula:

$$x \in P_{\omega\alpha+n} \longleftrightarrow h(x) \geq \omega\alpha \;\&\; px = 0 \;\&$$
$$\bigwedge_{k<n} \bigwedge_{\mathbf{r}^k} \text{not } h\Big(x - \sum_{\substack{q \in \mathbf{p}^k \\ r \in \mathbf{r}^k}} rq \Big) \geq \omega\alpha+k+1$$

The inner conjunction is taken over all sequences $\mathbf{r}^k$ corresponding in length to $\mathbf{p}^k$ with $0 \leq r < p$ for each element r and not all r zero. If

$$\bigwedge_{\mathbf{r}^k} \text{not } h\Big(x - \sum_{\substack{q \in \mathbf{p}^k \\ r \in \mathbf{r}^k}} rq\Big) \geq \omega\alpha+k+1$$

and $px=0$, then we have $h(x) \neq \omega\alpha+k$, as otherwise x must equal some linear combination of the $\mathbf{p}^k$ $(mod\ P_{\omega\alpha+k+1})$. So this together with

$h(x) \geq \omega\alpha$ implies that $x \in P_{\omega\alpha+n}$.

The converse is trivial as $h(\sum_{\substack{q \in \mathbf{p}_k^k \\ r \in \mathbf{r}^k}} rq) = \omega\alpha+k < \omega\alpha+n.$

So we have that $P_{\omega\alpha+n}$ is formally $\Pi^0_{2\alpha+1}$ and also formally $\Sigma^0_{2\alpha+1}$ from Proposition 5.1.1. Hence it is formally $\Delta^0_{2\alpha+1}$.

$P_{\omega\alpha+n}$ is not formally $\Sigma^0_{2\alpha}$ in the same way as $P_{\omega\alpha}$, and not formally $\Pi^0_{2\alpha}$ as follows:

Let $R = G - P_{\omega\alpha+n}$. Choose $r_1 \in P_{\omega\alpha+k} - P_{\omega\alpha+k+1}$ for some $k < n$. Given $\mathbf{a}$, choose $r' \in P_{\omega\alpha+n} - \langle \mathbf{a}, r_1 \rangle$, and let $r = r_1 + r'$.

Since $\mathbf{a}, r' \leq_{2\alpha} \mathbf{a}, r$, for any $\beta < \alpha$ and $\mathbf{c}$, there is $\mathbf{c}'$ with $\mathbf{a}, r, \mathbf{c} \leq_\beta \mathbf{a}, r', \mathbf{c}$. So $r \in R - Rcl_{2\alpha}(\mathbf{a})$.

(iii) To show that $P_{\omega\alpha+n}$ is not formally $\Pi^0_{2\alpha+1}$, we show that $G - P_{\omega\alpha+n}$ is not formally $\Sigma^0_{2\alpha+1}$.

Let $R = G - P_{\omega\alpha+n}$.

Given $\mathbf{a}$, choose $r \in P_{\omega\alpha+k} - P_{\omega\alpha+k+1} - \langle \mathbf{a} \rangle$ proper with respect to $\langle \mathbf{a} \rangle$. This is possible because $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$ is infinite, by Lemma 4.2.1. Choose $r' \in P_{\omega\alpha+n} - \langle \mathbf{a} \rangle$.

We claim that $r \in R - Rcl_{2\alpha+1}(\mathbf{a})$. Given $\mathbf{c}$, we want to find $\mathbf{c}'$ so that $\mathbf{a}, r, \mathbf{c} \leq_{2\alpha} \mathbf{a}, r', \mathbf{c}'$.

The fact that r is proper with respect to $\langle \mathbf{a} \rangle$ guarantees that $\mathbf{a}, r' \leq_{2\alpha+1} \mathbf{a}, r$, so this is immediate. □

Proposition 5.1.5: Suppose that $U_{\omega\alpha+n}(G) = \aleph_0$.

So $Q = \{ x : h(x) = \omega\alpha+n \ \& \ px = 0 \}$ is infinite. Then:

(i) If $U_{\omega\alpha+k}(G) = 0$ for each $k < n$ and $P_{\omega\alpha+n+1}$ is finite,

Q is formally $\Pi^0_{2\alpha}$, but not formally $\Sigma^0_{2\alpha}$.

(ii) If $U_{\omega\alpha+k}(G) < \aleph_0$ for each $k < n$ and $P_{\omega\alpha+n+1}$ is infinite,

Q is formally $\Pi^0_{2\alpha+1}$, but not formally $\Sigma^0_{2\alpha+1}$.

(iii) If $U_{\omega\alpha+k}(G) \neq 0$ for some $k < n$, $U_{\omega\alpha+k}(G) < \aleph_0$ for each $k < n$ and $P_{\omega\alpha+n+1}$ is finite,

Q is formally $\Delta^0_{2\alpha+1}$, but not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$.

(iv) If $U_{\omega\alpha+k}(G) = \aleph_0$ for some $k < n$ and $P_{\omega\alpha+n+1}$ is finite,

Q is formally $\Sigma^0_{2\alpha+1}$, but not formally $\Pi^0_{2\alpha+1}$.

(v) If $U_{\omega\alpha+k}(G) = \aleph_0$ for some $k < n$ and $P_{\omega\alpha+n+1}$ is infinite,

Q is formally $\Delta^0_{2\alpha+2}$, but not formally $\Sigma^0_{2\alpha+1}$ or $\Pi^0_{2\alpha+1}$.

Now suppose that $U_{\omega\alpha+n}(G) = m+1 < \aleph_0$ and $P_{\omega\alpha+n+1}(G)$ is infinite. Then again Q is infinite, and:

(vi) If $U_{\omega\alpha+k}(G) < \aleph_0$ for each $k < n$,

Q is formally $\Delta^0_{2\alpha+1}$, but not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$.

(vii) If $U_{\omega\alpha+k}(G) = \aleph_0$ for some $k < n$,

Q is formally $\Sigma^0_{2\alpha+1}$, but not formally $\Pi^0_{2\alpha+1}$.

Proof: (i) Let $\mathbf{p} = P_{\omega\alpha+n+1}$. Then

$$x \in Q \longleftrightarrow px = 0 \ \& \ h(x) \geq \omega\alpha \ \& \bigwedge_{q \in \mathbf{p}} x \neq q.$$

Q is not formally $\Sigma^0_{2\alpha}$ in the same way that $P_{\omega\alpha}$ is not formally $\Sigma^0_{2\alpha}$, as in Proposition 5.1.2.

(ii) Let $\mathbf{p}^k$ be sets of representatives of bases for $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$. Then we have the following $\Pi_{2\alpha+1}$ formula:

$$x \in Q \longleftrightarrow h(x) \geq \omega\alpha \ \& \ px = 0 \ \& \ \mathit{not}\ h(x) \geq \omega\alpha+n+1 \ \&$$

$$\bigwedge_{k<n} \bigwedge_{\mathbf{r}^k} \mathit{not}\ h\Big(x - \sum_{\substack{q \in \mathbf{p}^k_k \\ r \in \mathbf{r}^k}} rq \Big) \geq \omega\alpha+k+1.$$

This is similar to (ii) in the previous proposition.

To show that Q is not formally $\Sigma^0_{2\alpha+1}$, let $R = Q$.

Given $\mathbf{a}$, choose $r \in P_{\omega\alpha+n} - P_{\omega\alpha+n+1} - \langle\mathbf{a}\rangle$, proper with respect to $\langle\mathbf{a}\rangle$. We claim that $r \in R - Rcl_{2\alpha+1}(\mathbf{a})$.

Choose $r' \in P_{\omega\alpha+n+1} - \langle\mathbf{a}\rangle$. Then $\mathbf{a},r' \leq_{2\alpha+1} \mathbf{a},r$, so given $\mathbf{c}$ there is $\mathbf{c}'$ such that $\mathbf{a},r,\mathbf{c} \leq_{2\alpha} \mathbf{a},r',\mathbf{c}'$.

(iii) Let $\mathbf{p}^k$ be sets of representatives of bases for $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$, and let $\mathbf{p}^{n+1} = P_{\omega\alpha+n+1}$.

Then we have the following $\Pi_{2\alpha+1}$ formula:

$$x \in Q \longleftrightarrow h(x) \geq \omega\alpha \ \& \ px = 0 \ \& \ \mathit{not}\ h(x) \geq \omega\alpha+n+1 \ \&$$

$$\bigwedge_{k<n} \bigwedge_{\mathbf{r}^k} \mathit{not}\ h\Big(x - \sum_{\substack{q \in \mathbf{p}^k_k \\ r \in \mathbf{r}^k}} rq \Big) \geq \omega\alpha+k+1.$$

We also have the $\Sigma_{2\alpha+1}$ formula

$$x \in Q \longleftrightarrow px = 0 \ \& \ h(x) \geq \omega\alpha+n \ \& \bigwedge_{q \in \mathbf{p}^{n+1}} x \neq q.$$

But Q is not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$. Choose $r_1 \in P_{\omega\alpha+k} - P_{\omega\alpha+k+1}$. given $\mathbf{a}$, choose $r_2 \in P_{\omega\alpha+n} - P_{\omega\alpha+n+1} - \langle \mathbf{a}, r_1 \rangle$. Then we have $\mathbf{a}, r_1+r_2 \leq_{2\alpha} \mathbf{a}, r_2$ and $\mathbf{a}, r_1+r_2 \geq_{2\alpha} \mathbf{a}, r_2$.

(iv) Let $\mathbf{p} = P_{\omega\alpha+n+1}$. Then

$$x \in Q \longleftrightarrow px = 0 \ \& \ h(x) \geq \omega\alpha+n \ \& \bigwedge_{q \in \mathbf{p}} x \neq q.$$

So Q is formally $\Sigma_{2\alpha+1}$. Q is not formally $\Pi_{2\alpha+1}$ in the same way as in (iii) of the previous proposition.

(v) We have the $\Delta_{2\alpha+2}$ formula

$$x \in Q \longleftrightarrow px = 0 \ \& \ h(x) \geq \omega\alpha+n \ \& \ not\ h(x) \geq \omega\alpha+n+1.$$

Q is not formally $\Pi_{2\alpha+1}$ in the same way as in (iii) of the previous proposition, and not formally $\Sigma_{2\alpha+1}$ in the same way as in (ii) above.

(vi) Let $\mathbf{p}^k$ be sets of representatives of bases for $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$ for $k \leq n$. Then we have the $\Sigma_{2\alpha+1}$ formula:

$$x \in Q \longleftrightarrow \quad px = 0 \;\&\; h(x) \geq \omega\alpha{+}n \;\&$$

$$\bigvee_{\mathbf{r}^n} h\Big(x - \sum_{\substack{q \in \mathbf{p}^n \\ r \in \mathbf{r}^n}} rq \Big) \geq \omega\alpha{+}n{+}1.$$

We also have the $\Pi_{2\alpha+1}$ formula:

$$x \in Q \longleftrightarrow \quad h(x) \geq \omega\alpha \;\&\; px = 0 \;\&\; not\ h(x) \geq \omega\alpha{+}n{+}1 \;\&$$

$$\bigwedge_{k<n} \bigwedge_{\mathbf{r}^k} not\ h\Big(x - \sum_{\substack{q \in \mathbf{p}^k \\ r \in \mathbf{r}^k}} rq \Big) \geq \omega\alpha{+}k{+}1.$$

But Q is not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$. Choose $r_1 \in P_{\omega\alpha+n} - P_{\omega\alpha+n+1}$. given $\mathbf{a}$, choose $r_2 \in P_{\omega\alpha+n+1} - \langle \mathbf{a}, r_1 \rangle$. Then we have $\mathbf{a}, r_1 + r_2 \leq_{2\alpha} \mathbf{a}, r_2$ and $\mathbf{a}, r_1 + r_2 \geq_{2\alpha} \mathbf{a}, r_2$.

(vii) We have the $\Sigma_{2\alpha+1}$ formula of (vi) above, but Q is not formally $\Pi_{2\alpha+1}$ in the same way as in (iii) of the previous proposition. □

Proposition 5.1.6: Suppose, for some $n \geq 0$,

$$U_{\omega\alpha+k}(G) = 0 \quad \text{for each } 0 \leq k < n$$

$$U_{\omega\alpha+n}(G) = \aleph_0$$

$$U_{\omega\alpha+n+1}(G) < \aleph_0 \quad \text{and}$$

$$\lambda(G) \leq \omega\alpha{+}n{+}2.$$

Then (i) $G_{\omega\alpha+N}$ is formally $\Pi^0_{2\alpha}$ but not formally $\Sigma^0_{2\alpha}$, where $N \leq n$.
(ii) $Q = \{\ x \in G : h(x) = \omega\alpha{+}N\ \}$ is formally $\Pi^0_{2\alpha}$ but not formally $\Sigma^0_{2\alpha}$, where $N \leq n$.

Proof: (i) Let $\mathbf{p} = P_{\omega\alpha+n+1} = G_{\omega\alpha+n+1}$. These sets are equal and finite because of the conditions above: there are no non-zero elements of height $> \omega\alpha+n+1$. Then

$$h(x) \geq \omega\alpha+N \longleftrightarrow h(x) \geq \omega\alpha \ \& \bigvee_{q\in\mathbf{p}} p^{n-N+1}x = q$$

which is a $\Pi_{2\alpha}$ formula. To check this, first suppose $h(x) \geq \omega\alpha+N$. Then $h(p^{n-N+1}) \geq \omega\alpha+N + n-N+1 = \omega\alpha+n+1$, so $p^{n-N+1}x = q$ for some $q \in \mathbf{p}$.

Conversely, suppose $h(x) \geq \omega\alpha \ \& \bigvee_{q\in\mathbf{p}} p^{n-N+1}x = q$, but $h(x) = \omega\alpha+k < \omega\alpha+N$. From the second conjunct we have that $p^{l+1}x = 0$ for some $l \leq n-N+1$.

Let m be minimal such that $h(p^{m+1}x) > h(x) + m+1$.

If $m = n-N+1$, then $h(p^{n-N+1}x) = h(x) + n-N+1 < \omega\alpha+n+1$ which contradicts the second conjunct. So we have $m \leq n-N$.

Now $h(p^{m+1}x) > h(x) + m+1$, so there is y with $h(y) > h(x)$ and $p^{m+1}y = p^{m+1}x$. But then $p(p^m x-p^m y) = 0$ and

$$h(p^m x-p^m y) = h(p^m x) = h(x)+m = \omega\alpha+k+m \leq \omega\alpha+k+n-N < \omega\alpha+n.$$

But $h(p^m x-p^m y) \geq \omega\alpha$, which contradicts the Ulm invariants. This contradiction shows that $h(x) \geq \omega\alpha+N$.

Now we show that $R = G_{\omega\alpha+N}$ is not formally $\Sigma^0_{2\alpha}$.

This is similar to the argument above in Proposition 5.1.2. Thus if $\alpha = \beta+1$:

Suppose **a** is given. Choose $s \in P_{\omega\alpha+n} - \langle \mathbf{a} \rangle$ and r with $h(r) = \omega\alpha+N$ and $p^{n-N}r = s$.

Then, given **c**, choose $s' \in P_{\omega\beta+M} - P_{\omega\beta+M+1} - \langle \mathbf{a} \rangle$ for a sufficiently large M, and r' with $p^{n-N}r' = s'$ and $h(r') = \omega\beta+M-n+N$.

This shows that $r \in R - Rcl_{2\alpha}(\mathbf{a})$ as required.

(ii) Since

$$\text{not } h(x) \geq \omega\alpha+N+1 \longleftrightarrow h(x) < \omega\alpha \text{ or } \bigwedge_{r \in \mathbf{p}} p^{n-N}x \neq r$$

we have

$$h(x) = \omega\alpha+N \longleftrightarrow h(x) \geq \omega\alpha \;\&\; \bigvee_{r \in \mathbf{p}} p^{n-N+1}x = r \;\&\; \bigwedge_{r \in \mathbf{p}} p^{n-N}x \neq r$$

Q is not formally $\Sigma^0_{2\alpha}$ by the same argument as for $G_{\omega\alpha+N}$ in (i) above. □

Combining the ideas in the last two propositions, we have

Proposition 5.1.7: Suppose, for some $n > n' \geq 0$,

$$U_{\omega\alpha+k}(G) < \aleph_0 \text{ for each } 0 \leq k < n$$

$$U_{\omega\alpha+k}(G) \neq 0 \quad \text{for } k = n'$$

$$U_{\omega\alpha+n}(G) = \aleph_0$$

$$U_{\omega\alpha+n+1}(G) < \aleph_0 \quad \text{and}$$

$$\lambda(G) \leq \omega\alpha+n+2.$$

Then (i) $G_{\omega\alpha+N}$ is formally $\Delta^0_{2\alpha+1}$ but not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$, for $0 < N \leq n$.

(ii) $Q = \{ x \in G : h(x) = \omega\alpha+N \}$ is formally $\Delta^0_{2\alpha+1}$ but not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$, for $0 \leq N \leq n$.

Proof: (i) Let $\mathbf{p}^k$ be sets of representatives of bases for $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$, and let $\mathbf{p}^{n+1} = P_{\omega\alpha+n+1}$.

We know that $G_{\omega\alpha+N}$ is formally $\Sigma^0_{2\alpha+1}$. The following formula shows $G_{\omega\alpha+N}$ is formally $\Pi^0_{2\alpha+1}$.

$$h(x) \geq \omega\alpha+N \quad \longleftrightarrow \quad h(x) \geq \omega\alpha \quad \& \quad \bigvee_{q\in\mathbf{p}^{n+1}} p^{n-N+1}x = q \quad \&$$

$$\bigwedge_{k<N} \ \bigwedge_{m<n-k} \ \bigwedge_{\mathbf{r}^{k+m}} \ not\ h\Big(p^m x - \sum_{\substack{q\in\mathbf{p}^{k+m} \\ r\in\mathbf{r}^{k+m}}} rq \Big) \geq \omega\alpha+k+m+1.$$

The inner conjunction is taken over all sequences $\mathbf{r}^{k+m}$ corresponding in length to $\mathbf{p}^{k+m}$ with $0 \leq r < p$ for each element r and not all r zero.

To check this, first suppose that $h(x) \geq \omega\alpha+N$. Then $h(p^{n-N+1}x) \geq \omega\alpha+n+1$, so $p^{n-N+1}x = q$ for some $q \in \mathbf{p}^{n+1}$. Also $h(p^m x) \geq \omega\alpha+N+m > \omega\alpha+k+m$ for $k < N$. So

$$h\Big(p^m x - \sum_{\substack{q\in\mathbf{p}^{k+m} \\ r\in\mathbf{r}^{k+m}}} rq \Big) = \omega\alpha+k+m$$

and we see that every conjunct of the third condition is satisfied.

Conversely suppose that

$$h(x) \geq \omega\alpha \quad \& \quad \bigvee_{q \in \mathbf{p}^{n+1}} p^{n-N+1}x = q \quad \&$$

$$\bigwedge_{k<N} \bigwedge_{m<n-k} \bigwedge_{\mathbf{r}^{k+m}} \text{not } h\Big(p^{m}x - \sum_{\substack{q \in \mathbf{p}^{k+m} \\ r \in \mathbf{r}^{k+m}}} rq \Big) \geq \omega\alpha+k+m+1.$$

but $h(x) = \omega\alpha+k < \omega\alpha+N$.

Then for some $l \leq n-N+1$, we have $p^{l+1}x = 0$.

Let m be minimal such that $h(p^{m+1}x) > h(x) + m+1$.

If $m = n-N+1$, then $h(p^{n-N+1}x) = h(x) + n-N+1 < \omega\alpha+n+1$ which contradicts the second conjunct. So we have $m \leq n-N < n-k$.

So there is a conjunct $\bigwedge_{\mathbf{r}^{k+m}} \text{not } h\big(p^{m}x - \sum_{\substack{q \in \mathbf{p}^{k+m} \\ r \in \mathbf{r}^{k+m}}} rq \big) \geq \omega\alpha+k+m+1$ in the assumed condition. Now $h(p^{m+1}x) > h(x) + m+1$, so there is y with $h(y) > h(x)$ and $p^{m+1}y = p^{m+1}x$. But then $p(p^{m}x-p^{m}y) = 0$ and $h(p^{m}x-p^{m}y) = h(p^{m}x) = h(x)+m = \omega\alpha+k+m$. So there is a sequence $\mathbf{r}^{k+m}$ and some $q' \in P_{\omega\alpha+k+m+1}$ such that $(p^{m}x-p^{m}y) - \sum_{\substack{q \in \mathbf{p}^{k+m} \\ r \in \mathbf{r}^{k+m}}} rq = q'$, by the definition of $\mathbf{p}^{k+m}$. But then $h\big(p^{m}x - \sum_{\substack{q \in \mathbf{p}^{k+m} \\ r \in \mathbf{r}^{k+m}}} rq \big) = h(p^{m}y + q')$ $\geq \omega\alpha+k+m+1$, which contradicts the above conjunct. So we must have $h(x) \geq \omega\alpha+N$.

$G_{\omega\alpha+N}$ is not formally $\Sigma^0_{2\alpha}$ by the same argument as in the previous proposition. If $N > 0$, we show $G_{\omega\alpha+N}$ is not formally $\Pi^0_{2\alpha}$ by showing that $R = G - G_{\omega\alpha+N}$ is not formally $\Sigma^0_{2\alpha}$.

Choose $r_1 \in P_{\omega\alpha+n'} - P_{\omega\alpha+N}$. Given $\mathbf{a}$, choose $r' \in P_{\omega\alpha+n} - \langle\mathbf{a}\rangle$. Then, if $N > n'$, $r_1 + r' \in R - Rcl_{2\alpha}(\mathbf{a})$, because $\mathbf{a},r' \leq_{2\alpha} \mathbf{a},r_1+r'$.

For $0 < N \leq n'$, choose $s_1 \in G_{\omega\alpha+N-1}$ with $p^{n'-N+1}s_1 = r_1$ and $s' \in G_{\omega\alpha+n-n'+N-1}$ with $p^{n'-N+1}s' = r'$. Then $s_1 + s' \in R - Rcl_{2\alpha}(\mathbf{a})$, because $\mathbf{a}, s' \leq_{2\alpha} \mathbf{a}, s_1 + s'$.

(ii) We have the $\Pi^0_{2\alpha+1}$ formula

$$h(x) = \omega\alpha+N \quad \longleftrightarrow \quad h(x) \geq \omega\alpha \;\;\&\; \bigvee_{q\in \mathbf{p}^{n+1}} p^{n-N+1}x = q \;\;\&$$

$$\bigwedge_{k<N} \; \bigwedge_{m<n-k} \; \bigwedge_{\mathbf{r}^{k+m}} \text{not } h\Big(p^m x - \sum_{\substack{q\in \mathbf{p}_{k+m} \\ r\in \mathbf{r}^{k+m}}} rq \Big) \geq \omega\alpha+k+m+1.$$

$$\& \text{ not } h(x) \geq \omega\alpha+N+1.$$

Since

$$\text{not } h(x) \geq \omega\alpha+N+1 \quad \longleftrightarrow \quad h(x) < \omega\alpha \;\text{ or }\; \bigwedge_{q\in \mathbf{p}^{n+1}} p^{n-N}x \neq q \;\text{ or}$$

$$\bigvee_{k<N+1} \; \bigvee_{m<n-k} \; \bigvee_{\mathbf{r}^{k+m}} h\Big(p^m x - \sum_{\substack{q\in \mathbf{p}_{k+m} \\ r\in \mathbf{r}^{k+m}}} rq \Big) \geq \omega\alpha+k+m+1.$$

we have the $\Sigma^0_{2\alpha+1}$ formula

$$h(x) = \omega\alpha+N \quad \longleftrightarrow \quad h(x) \geq \omega\alpha+N \;\;\&\; \Big[\bigwedge_{q\in \mathbf{p}^{n+1}} p^{n-N}x \neq q \;\text{ or}$$

$$\bigvee_{k<N+1} \; \bigvee_{m<n-k} \; \bigvee_{\mathbf{r}^{k+m}} h\Big(p^m x - \sum_{\substack{q\in \mathbf{p}_{k+m} \\ r\in \mathbf{r}^{k+m}}} rq \Big) \geq \omega\alpha+k+m+1 \Big].$$

Now we show that Q is not formally $\Sigma^0_{2\alpha}$ or $\Pi^0_{2\alpha}$. First suppose that $N \leq n'$. Choose $s_1 \in P_{\omega\alpha+n'} - P_{\omega\alpha+n'+1}$ and choose r_1 such that

$h(r_1) = \omega\alpha+N$ and $p^{n'-N}r_1 = s_1$. Given **a**, choose $s' \in P_{\omega\alpha+n} - \langle a \rangle$ and choose r' such that $h(r') > \omega\alpha+N$ and $p^{n'-N}r' = s'$.

Then $\mathbf{a},r_1+r' \leq_{2\alpha} \mathbf{a},r'$ and $\mathbf{a},r' \leq_{2\alpha} \mathbf{a},r_1+r'$.

Now suppose that $N > n'$. Choose $r' \in P_{\omega\alpha+n'} - P_{\omega\alpha+N}$. Given **a**, choose $s \in P_{\omega\alpha+n} - \langle \mathbf{a} \rangle$ and choose r such that $h(r) = \omega\alpha+N$ and $p^{n-N}r = s$. Then $\mathbf{a},r \leq_{2\alpha} \mathbf{a},r+r'$ and $\mathbf{a},r+r' \leq_{2\alpha} \mathbf{a},r$. □

Proposition 5.1.8: Suppose for some n, $U_{\omega\alpha+n}(G) = \aleph_0$ and either

$P_{\omega\alpha+n+2}$ is infinite ~~$\neq \{0\}$~~ or $U_{\omega\alpha+n+1}(G) = \aleph_0$.

Then (i) $G_{\omega\alpha+N}$ is formally $\Sigma^0_{2\alpha+1}$ but not formally $\Pi^0_{2\alpha+1}$, for $0 < N \leq n$.

(ii) $Q = \{ x : h(x) = \omega\alpha+N \}$ is formally $\Delta^0_{2\alpha+2}$ but not formally $\Sigma^0_{2\alpha+1}$ or $\Pi^0_{2\alpha+1}$, for $0 < N \leq n$.

For $N = 0$, Q is formally $\Pi^0_{2\alpha+1}$ but not formally $\Sigma^0_{2\alpha+1}$.

For $N = n$, if $P_{\omega\alpha+n+1}$ is finite, Q is formally $\Sigma^0_{2\alpha+1}$ but not formally $\Pi^0_{2\alpha+1}$.

Proof: Case 1: $P_{\omega\alpha+n+2}$ is infinite. ~~$\neq \{0\}$~~.

(i) To show that $G_{\omega\alpha+N}$ is not formally $\Pi^0_{2\alpha+1}$, consider $R = G - G_{\omega\alpha+N}$.

Given $\bar{a}$, Choose $s \in P_{\omega\alpha+n+2} - \langle \bar{a} \rangle$ ~~$\{0\}$~~, $s_1 \in G_{\omega\alpha+n+1}$ such that $ps_1 = s$, and $r_1 \in G_{\omega\alpha+N}$ such that $p^{n-N+1}r_1 = s_1$.

~~Given a,~~ choose $s_2 \in P_{\omega\alpha+n} - P_{\omega\alpha+n+1} - \langle \mathbf{a},r_1 \rangle$ proper with respect to $\langle \mathbf{a},r_1 \rangle$, and choose r_2 such that $h(r_2) = \omega\alpha+N-1$ and $p^{n-N+1}r_2 = s_2$. Then r_2 must also be proper with respect to $\langle \mathbf{a},r_1 \rangle$.

[Suppose $h(r_2+ a) > h(r_2) = \omega\alpha+N-1$. Then

$$h(\ p^{n-N+1}(r_2+a)\ ) > \omega\alpha+N-1 + n-N+1 = \omega\alpha+n.$$

That is $h(s_2+ a') > h(s_2)$, contradicting the properness of s_2.]

Also take $r_3 = pr_2$. Then $h(r_3) = \omega\alpha+N$ and r_3 is also proper with respect to $\langle \mathbf{a},r_1\rangle$.

Let $r = r_1+ r_2$. Then $h(r) = \omega\alpha+N-1$ and $p^{n-N+1}r = s_1+ s_2$.
Let $r' = r_1+ r_3$. Then $h(r') = \omega\alpha+N$ and $p^{n-N+1}r' = s_1+ ps_2 = s_1$.

Then $\mathbf{a},s_1,r' \le_{2\alpha+1} \mathbf{a},s_1+ s_2,r$, so given $\mathbf{b}$ we can find $\mathbf{b}'$ such that $\mathbf{a},r,\mathbf{b} \le_{2\alpha} \mathbf{a},r',\mathbf{b}'$. Hence $r \in R - Rcl_{2\alpha+1}(\mathbf{a})$.

(ii) The above argument also shows that Q is not formally $\Pi^0_{2\alpha+1}$, for $0 < N \le n$. Also, reversing the roles of r and r', it shows that Q is not formally $\Sigma^0_{2\alpha+1}$, for $0 \le N < n$.

For $0 < N < n$, we have the $\Delta^0_{2\alpha+2}$ formula

$$x \in Q \longleftrightarrow h(x) \ge \omega\alpha+N \ \&\ not\ h(x) \ge \omega\alpha+N+1$$

For $N = 0$, this formula is $\Pi^0_{2\alpha+1}$.
For $N = n$, if $P_{\omega\alpha+n+1}$ is finite, say $\mathbf{p} = P_{\omega\alpha+n+1}$, we have the $\Sigma^0_{2\alpha+1}$ formula

$$x \in Q \longleftrightarrow h(x) \ge \omega\alpha+n \ \&\ \bigwedge_{q\in\mathbf{p}} x \ne q$$

Case 2. $U_{\omega\alpha+n+1}(G) = \aleph_0$.

(i) To show that $G_{\omega\alpha+N}$ is not formally $\Pi^0_{2\alpha+1}$, consider $R = G - G_{\omega\alpha+N}$.

Given $\mathbf{a}$, choose $s \in P_{\omega\alpha+n} - P_{\omega\alpha+n+1} - \langle\mathbf{a}\rangle$ proper with respect to $\langle\mathbf{a}\rangle$, and choose r such that $h(r) = \omega\alpha+N-1$ and $p^{n-N+1}r = s$. Then

r must also be proper with respect to $\langle \mathbf{a} \rangle$.

Choose $s' \in P_{\omega\alpha+n+1} - P_{\omega\alpha+n+2} - \langle \mathbf{a} \rangle$, and r' such that $h(r') = \omega\alpha+N$ and $p^{n-N+1}r' = s'$. Then $\mathbf{a},r' \leq_{2\alpha+1} \mathbf{a},r$ so $r \in R - Rcl_{2\alpha+1}(\mathbf{a})$.

(ii) This is exactly the same as (ii) for case 1 above. □

Proposition 5.1.9: Suppose $\lambda(G) \geq \omega\alpha+\omega$.

Then (i) $G_{\omega\alpha+N}$ is formally $\Sigma^0_{2\alpha+1}$ but not formally $\Pi^0_{2\alpha+1}$, for $N > 0$.

(ii) $Q = \{ x : h(x) = \omega\alpha+N \}$ is formally $\Delta^0_{2\alpha+2}$ but not formally $\Sigma^0_{2\alpha+1}$ or $\Pi^0_{2\alpha+1}$, for $N > 0$.

For $N = 0$, Q is formally $\Pi^0_{2\alpha+1}$ but not formally $\Sigma^0_{2\alpha+1}$.

Proof: (i) This is similar to case 1 of the previous proposition.

To show that $G_{\omega\alpha+N}$ is not formally $\Pi^0_{2\alpha+1}$, consider $R = G - G_{\omega\alpha+N}$.
Given $\mathbf{a}$, choose $n \geq N$ so that $U_{\omega\alpha+n}(G) \neq 0$ and $\langle \mathbf{a} \rangle$ contains no elements of height $\omega\alpha+n$.

Choose $s \in P_{\omega\alpha+n+2} - \overset{\langle a \rangle}{\cancel{\{0\}}}$, $s_1 \in G_{\omega\alpha+n+1}$ such that $ps_1 = s$, and $r_1 \in G_{\omega\alpha+N}$ such that $p^{n-N+1}r_1 = s_1$.

~~Given a,~~ choose $s_2 \in P_{\omega\alpha+n} - P_{\omega\alpha+n+1} - \langle \mathbf{a},r_1 \rangle$ proper with respect to $\langle \mathbf{a},r_1 \rangle$, and choose r_2 such that $h(r_2) = \omega\alpha+N-1$ and $p^{n-N+1}r_2 = s_2$.
Then r_2 must also be proper with respect to $\langle \mathbf{a},r_1 \rangle$.

Also take $r_3 = pr_2$. Then $h(r_3) = \omega\alpha+N$ and r_3 is also proper with respect to $\langle \mathbf{a},r_1 \rangle$.

Let $r = r_1 + r_2$. Then $h(r) = \omega\alpha+N-1$ and $p^{n-N+1}r = s_1 + s_2$.
Let $r' = r_1 + r_3$. Then $h(r') = \omega\alpha+N$ and $p^{n-N+1}r' = s_1 + ps_2 = s_1$.

Then $\mathbf{a},s_1,r' \le_{2\alpha+1} \mathbf{a},s_1+s_2,r$, so given $\mathbf{b}$ we can find $\mathbf{b}'$ such that $\mathbf{a},r,\mathbf{b} \le_{2\alpha} \mathbf{a},r',\mathbf{b}'$. Hence $r \in R - Rcl_{2\alpha+1}(\mathbf{a})$.

(ii) As in (ii) in the previous proposition, the above argument also shows that Q is not formally $\Pi^0_{2\alpha+1}$, for $N > 0$. And, reversing the roles of r and r', it shows that Q is not formally $\Sigma^0_{2\alpha+1}$, for $N \ge 0$.

For $N > 0$, we have the $\Delta^0_{2\alpha+2}$ formula

$$x \in Q \longleftrightarrow h(x) \ge \omega\alpha+N \;\&\; not\; h(x) \ge \omega\alpha+N+1$$

For $N = 0$, this formula is $\Pi^0_{2\alpha+1}$. □

In each case where we have a statement that a given set is not formally Σ^0_α or Π^0_α, we can apply Theorem 2.7.1, using Propositions 4.2.2 and 4.3.3, to conclude that the set is not intrinsically Σ^0_α or Π^0_α. That is to say that there is an isomorphic recursive abelian p-group on which the corresponding set is not Σ^0_α or Π^0_α.

§5.2 Δ^0_α Categoricity.

We say that a recursive structure $\mathfrak{A}$ is Δ^0_α-*categorical*, for $\alpha < \omega_1^{CK}$, if for every recursive structure $\mathfrak{B} \cong \mathfrak{A}$ there exists an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ which is a Δ^0_α function.

In [A3], Ash gives a syntactic condition which, in the presence of some decidability assumptions, is equivalent to Δ^0_α-categoricity. Here we apply this to the case of reduced abelian p-groups.

There is an omission in the statement of the theorem in [A3] which we will demonstrate using the example of a reduced abelian p-group. The modifications required in the statement and proof of the result are very minor.

Definition 5.2.1: A Σ^0_α *Scott family* for the recursive structure $\mathfrak{A}$ is a Σ^0_α set of Gödel numbers for recursive Σ_α formulae $\phi_n(\mathbf{x},\mathbf{y})$ together with a finite sequence $\mathbf{p}$ from A, with the following properties:

(a) For each $\mathbf{a} \in A$, there exists n such that $\mathfrak{A} \models \phi_n(\mathbf{p},\mathbf{a})$.

(b) For each n and each $\mathbf{a},\mathbf{b} \in A$, if $\mathfrak{A} \models \phi_n(\mathbf{p},\mathbf{a})$ and $\mathfrak{A} \models \phi_n(\mathbf{p},\mathbf{a})$ then $(\mathfrak{A},\mathbf{a}) \cong (\mathfrak{B},\mathbf{b})$.

Definition 5.2.2: For $\mathbf{a} \in A$, $2 \leq \alpha$, let $C_\alpha(\mathbf{a})$ denote the set of sequences $\mathbf{c} \in A$ for which there exists $\mathbf{b} \in A$ and $\beta < \alpha$ such that

$$\mathbf{a,c,b} \leq_\beta \mathbf{a,c',b'} \implies \mathbf{a,c} \geq_\alpha \mathbf{a,c'} \;\;\&\;\; \mathbf{a,c} \leq_\alpha \mathbf{a,c'}$$

for every $\mathbf{b',c'} \in A$.

Ash's result on Δ^0_α-categoricity is the following:

Theorem 5.2.3: Let $2 \leq \alpha < \omega_1^{CK}$ and let $\mathfrak{A}$ be a recursive structure satisfying the following conditions:

(A) The existential diagram of $\mathfrak{A}$ is recursive.

(B) The relations $\leq_\gamma$ for $\gamma < \alpha$ are recursively enumerable, uniformly in γ.

(C) The relation $\mathbf{c} \notin C_\alpha(\mathbf{a})$ is recursively enumerable.

(D) The relation (*not* $\leq_\alpha$) is recursively enumerable.

Then $\mathfrak{A}$ is Δ^0_α-categorical iff $\mathfrak{A}$ has a Σ^0_α Scott family. ■

One of the basic ideas of the proof is the following lemma:

Lemma 5.2.4: Under the conditions of Theorem 5.2.3, if there exists $\mathbf{p} \in A$ such that, for all $\mathbf{c}$, $\mathbf{c} \in C_\alpha(\mathbf{p})$, then $\mathfrak{A}$ has a Σ^0_α Scott family.

Ash's definition of $C_\alpha(\mathbf{a})$ was simply:

Definition': For $\mathbf{a} \in A$, $2 \leq \alpha$, let $C'_\alpha(\mathbf{a})$ denote the set of sequences $\mathbf{c} \in A$ for which there exists $\mathbf{b} \in A$ and $\beta < \alpha$ such that

$$\mathbf{a,c,b} \leq_\beta \mathbf{a,c',b'} \implies \mathbf{a,c} \geq_\alpha \mathbf{a,c'}$$

for every $\mathbf{b',c'} \in A$.

For this definition the proof of the above lemma does not go through. We demonstrate this by exhibiting a reduced abelian p-group $\mathfrak{G}$ which is not Δ^0_3 categorical, and so has no Σ^0_3 Scott family, but for which $C'_3(-) = G$ (where - is the empty sequence).

Let $\mathfrak{G}$ be a recursive copy with recursive height function of the reduced abelian p-group with length $\lambda(\mathfrak{G}) = \omega 2$ and Ulm invariants $U_\beta(\mathfrak{G}) = 1$ for every $\beta < \omega 2$.

In our general discussion of Δ^0_α categoricity for reduced abelian p-groups we will see that $\mathfrak{G}$ is not Δ^0_3 categorical (Proposition 5.2.6, case 1 with $\alpha = 1$).

Proposition 5.2.5: $C'_3(-) = G$.

Proof: Given any $\mathbf{c} \in G$, we need to find $\mathbf{b}$ so that

$$\mathbf{c,b} \leq_2 \mathbf{c',b'} \implies \mathbf{c} \geq_3 \mathbf{c'} \quad \text{for every } \mathbf{b',c'} \in G.$$

For each $c \in \langle\mathbf{c}\rangle$, if $h(c) = \omega+n$, choose $b \in G_\omega$ with $p^n b = c$.
Then $\mathbf{c,b} \leq_2 \mathbf{c',b'}$ implies $h(b),h(b') \geq \omega$ *for each* $b \in \mathbf{b}$ and corresponding b'. This implies that $h(c') \geq h(c)$ for each $c \in \mathbf{c}$ with $h(c) \geq \omega$ and corresponding c'. Also we must have $h(c) = h(c')$ for each c with $h(c) < \omega$. Hence we have $\mathbf{c} \geq_3 \mathbf{c'}$. □

Under the correct definition of $C_\alpha(\mathbf{a})$ the only alteration that is required in Ash's proof is the observation that if a one-one function does not preserve Π^0_α formulae then its inverse does not preserve Σ^0_α formulae, and vice versa.

Now we give the general result on Δ^0_α categoricity for countable reduced abelian p-groups with recursive Ulm invariants.

We say $\mathfrak{A}$ is $\hat{\Delta}^0_\alpha$ categorical if for each recursive $\mathfrak{B} \cong \mathfrak{A}$ there is an isomorphism which is Δ^0_β for some $\beta < \alpha$. It is shown in [A3] that under the conditions of the above theorem a recursive structure is $\hat{\Delta}^0_\alpha$ categorical iff it is Δ^0_β categorical for some $\beta < \alpha$.

Proposition 5.2.6: Let G be a countable reduced abelian p-group with recursive Ulm invariants such that:

1. $\lambda(G) = \omega\alpha+\omega+n$ and $G_{\omega\alpha+\omega}$ is finite.

Then G is $\Delta^0_{2\alpha+2}$ categorical but not $\Delta^0_{2\alpha+1}$ categorical.

2. $\lambda(G) \leq \omega\alpha+n+2$ and

$$\begin{aligned} U_{\omega\alpha+k}(G) &< \aleph_0, \quad 0 \leq k < n \\ &= \aleph_0, \quad k = n \\ &< \aleph_0, \quad k = n+1. \end{aligned}$$

Then G is $\Delta^0_{2\alpha+1}$ categorical but not $\Delta^0_{2\alpha}$ categorical.

3. $\lambda(G) = \omega\alpha+n+1$ and

$$U_{\omega\alpha}(G) = \aleph_0 \text{ and}$$

$$U_{\omega\alpha+k}(G) < \aleph_0 \text{ for } 1 \leq k \leq n.$$

Then G is $\Delta^0_{2\alpha+1}$ categorical but not $\Delta^0_{2\alpha}$ categorical.

4. $\lambda(G) = \omega\alpha+n+1$ and

$$U_{\omega\alpha+N}(G) = \aleph_0 \text{ for some } 1 \leq N < n \text{ and}$$

$G_{\omega\alpha+N+2}$ is infinite.

$$U_{\omega\alpha+N+1}(G) = \aleph_0 \text{ or } \cancel{U_{\omega\alpha+k}(G) \neq 0 \text{ for some } k \geq N+2.}$$

Then G is $\Delta^0_{2\alpha+2}$ categorical but not $\Delta^0_{2\alpha+1}$ categorical.

5. $\lambda(G) = \omega\alpha+n$ for limit ordinal α, and $G_{\omega\alpha}$ is finite.

Then G is Δ^0_{α} categorical but not $\hat{\Delta}^0_{\alpha}$ categorical.

Proof: Propositions 4.2.2 and 4.3.3 show that the required conditions of Theorem 5.2.3 are satisfied.

In each case it is easy to demonstrate the categoricity result by exhibiting an appropriate Scott family, using the results of the previous section.

From the definition of a Scott family, we need formulae such that every finite sequence **a** from G satisfies some formula of the family, and if **a** and **b** satisfy the same formula then $(G,\mathbf{a})\cong(G,\mathbf{b})$. That is to say, the map of corresponding elements of the sequence $f: \mathbf{a} \longrightarrow \mathbf{b}$ must extend to an isomorphism $f: \langle\mathbf{a}\rangle \cong \langle\mathbf{b}\rangle$, which must itself be extendible to an automorphism of G. But by Ulm's Theorem this will occur when $\langle\mathbf{a}\rangle$ and $\langle\mathbf{b}\rangle$ satisfy the same group addition table and corresponding elements have the same height.

Given a finite sequence of elements **a**, let **x** be a corresponding sequence of variables. Let $S_{\mathbf{a}}(\mathbf{x})$ be a formula completely specifying the group addition table of $\langle \mathbf{a} \rangle$.

Now we have the following formulae, repeated from Proposition 5.1.1:

$$x \in G_{n} \underset{df}{\longleftrightarrow} \exists y \;(p^{n}y = x)$$

$$x \in G_{\omega} \underset{df}{\longleftrightarrow} \bigwedge\!\!\bigwedge_{n<\omega} \exists y \;(p^{n}y = x)$$

$$x \in G_{\omega\alpha+n} \underset{df}{\longleftrightarrow} \exists y \;(p^{n}y = x \;\&\; y \in G_{\omega\alpha})$$

$$x \in G_{\omega\alpha+\omega} \underset{df}{\longleftrightarrow} \bigwedge\!\!\bigwedge_{n<\omega} \exists y \;(p^{n}y = x \;\&\; y \in G_{\omega\alpha})$$

$$x \in G_{\omega\alpha} \underset{df}{\longleftrightarrow} \bigwedge\!\!\bigwedge_{\gamma<\alpha} (x \in G_{\omega\gamma}) \qquad \text{for limit ordinals } \alpha$$

By induction we see that $x \in G_{\omega\alpha}$ is a recursive $\Pi_{2\alpha}$ formula and $x \in G_{\omega\alpha+n}$ is a recursive $\Sigma_{2\alpha+1}$ formula. Thus we have the following:

$h(x) = \omega\alpha$ can be expressed by the recursive $\Pi_{2\alpha+1}$ formula $x \in G_{\omega\alpha} \;\&\; x \notin G_{\omega\alpha+1}$.

$h(x) \geq \omega\alpha$ can be expressed by the recursive $\Pi_{2\alpha}$ formula $x \in G_{\omega\alpha}$.

$h(x) = \omega\alpha+n$ can be expressed by the recursive $\Delta_{2\alpha+2}$ formula $x \in G_{\omega\alpha+n} \;\&\; x \notin G_{\omega\alpha+n+1}$.

Let $\psi_{\mathbf{a}}(\mathbf{x})$ be the formula

$$S_{\mathbf{a}}(\mathbf{x}) \;\&\; \bigwedge_{a\in\langle \mathbf{a}\rangle} \text{'}x \text{ has height } h_{a}\text{'}$$

where the way *'x has height* h_a*'* is expressed using the above formulae will be specified below in each case.

Case 1. Since $G_{\omega\alpha+\omega}$ is finite, let $\mathbf{p} = G_{\omega\alpha+\omega}$. Then

$\{ \psi_{\mathbf{a}}$: **a** *is a finite sequence of elements from G* $\}$

is a $\Sigma^0_{2\alpha+2}$ Scott family for G, where *'x has height h_a'* means

$$\begin{cases} h(x) = h_a & \text{for } h_a < \omega\alpha+\omega \\ x = p \ \text{ for the } p = a \in \mathbf{p} & \text{for } h_a \geq \omega\alpha+\omega \end{cases}$$

However G has no $\Sigma^0_{2\alpha+1}$ Scott family. To show this, we must show how, given **a**, to find $\mathbf{c} \notin C_{2\alpha+1}(\mathbf{a})$.

Choose $c \in P_{\omega\alpha+k} - P_{\omega\alpha+k+1} - \langle \mathbf{a},\mathbf{p}\rangle$, for some k, proper with respect to $\langle \mathbf{a},\mathbf{p}\rangle$.

We claim $c \notin C_{2\alpha+1}(\mathbf{a})$.

Choose $c' \in P_{\omega\alpha+k+1} - \langle \mathbf{a},\mathbf{p}\rangle$. Then *not* $\mathbf{a},c \leq_{2\alpha+1} \mathbf{a},c'$, but $\mathbf{a},c' \leq_{2\alpha+1} \mathbf{a},c$, so for any **b** we can find **b'** so that $\mathbf{a},c,\mathbf{b} \leq_{2\alpha} \mathbf{a},c',\mathbf{b}'$.

Case 2. We have that $G_{\omega\alpha+n+1} = P_{\omega\alpha+n+1}$ is finite, and $G_{\omega\alpha+n+2} = \{0\}$. Let $\mathbf{p}^k$ be sets of representatives of bases for $P_{\omega\alpha+k}/P_{\omega\alpha+k+1}$, and let $\mathbf{p}^{n+1} = P_{\omega\alpha+n+1}$. Then

$\{ \psi_{\mathbf{a}}$: **a** *is a finite sequence of elements from G* $\}$

is a $\Sigma^0_{2\alpha+1}$ Scott family for G, where *'x has height h_a'* means

$$\begin{cases} h(x) = h_a \quad \text{for} \quad h_a < \omega\alpha \\ h(x) \geq \omega\alpha+N \;\; \& \left[\bigvee_{k<N+1} \bigvee_{m<n-k} \bigvee_{\mathbf{r}^{k+m}} h(p^m x - \sum_{\substack{q\in\mathbf{p}^{k+m} \\ r\in\mathbf{r}^{k+m}}} rq) \geq \omega\alpha+k+m+1 \right. \\ \qquad\qquad \left. \text{or} \bigwedge_{q\in\mathbf{p}^{n+1}} p^{n-N}x \neq q \right] \quad \text{for } h(x) = \omega\alpha+N \leq \omega\alpha+n \\ x = p \text{ for the } p = a \in \mathbf{p}^{n+1} \quad \text{for } h_a = \omega\alpha+n+1 \end{cases}$$

Note that $0 \in \mathbf{p}^{n+1}$, which is needed in the formula for $h_a = \omega\alpha+N$.

However G has no $\Sigma^0_{2\alpha}$ Scott family.

Given $\mathbf{a}$, choose $c \in P_{\omega\alpha+n} - \langle\mathbf{a},\mathbf{p}\rangle$. We claim $c \notin C_{2\alpha}(\mathbf{a})$.
So, given $\mathbf{b}$ and $\beta < 2\alpha$, we want to find c' and $\mathbf{b}'$ so that
$\mathbf{a},c,\mathbf{b} \leq_\beta \mathbf{a},c',\mathbf{b}'$ and *not* $\mathbf{a},c \geq_{2\alpha} \mathbf{a},c'$.
If $\alpha = \gamma+1$, choose $c' \in P_{\omega\gamma+M} - P_{\omega\gamma+M+1} - \langle\mathbf{a},\mathbf{p}\rangle$, where

$$M \geq max\{m : b\in\langle\mathbf{a},\mathbf{p},c,\mathbf{b}\rangle,\ h(b)=\omega\gamma+m \text{ and } m<\omega\}$$
$$+ \ Card(\langle\mathbf{a},\mathbf{p},c,\mathbf{b}\rangle - \langle\mathbf{a},\mathbf{p}\rangle) + 1.$$

Then *not* $\mathbf{a},c \geq_{2\alpha} \mathbf{a},c'$, but we can find $\mathbf{b}'$ so that

$$\mathbf{a},c,\mathbf{b} \leq_{2\gamma+1} \mathbf{a},c',\mathbf{b}'.$$

[This is the same argument as in case **(Aib)** of Proposition 4.2.2.]
If α is a limit ordinal, choose $c' \in P_{\omega(\beta+1)} - P_{\omega(\beta+2)} - \langle\mathbf{a},\mathbf{p}\rangle$.

Case 3. We have that $P_{\omega\alpha+1}$ is finite, so $G_{\omega\alpha+1}$ is finite. Let $\mathbf{p} = G_{\omega\alpha+1}$. Then

$$\{\ \psi_{\mathbf{a}} : \mathbf{a} \text{ is a finite sequence of elements from } G\ \}$$

is a $\Sigma^0_{2\alpha+1}$ Scott family for G, where *'x has height h_a'* means

$$\begin{cases} h(x) = h_a \quad \text{for } h_a < \omega\alpha \\ h(x) \geq \omega\alpha \;\&\; \bigwedge_{p \in \mathbf{p}} x \neq p \quad \text{for } h_a = \omega\alpha \\ x = p \quad \text{for the } p = a \in \mathbf{p} \quad \text{for } h_a > \omega\alpha \end{cases}$$

However G has no $\Sigma^0_{2\alpha}$ Scott family. The argument is as in case 2, choosing $c \in P_{\omega\alpha} - \langle \mathbf{a},\mathbf{p} \rangle$.

Case 4. $\{\, \psi_{\mathbf{a}}$: **a** *is a finite sequence of elements from* $G \,\}$ is a $\Sigma^0_{2\alpha+2}$ Scott family for G, where *'x has height h_a'* means

$$h(x) = h_a.$$

However G has no $\Sigma^0_{2\alpha+1}$ Scott family. There are two cases to consider:

(a) $U_{\omega\alpha+N+1}(G) = \aleph_0$.

Given **a**, choose $c \in P_{\omega\alpha+N} - P_{\omega\alpha+N+1} - \langle \mathbf{a} \rangle$, proper with respect to $\langle \mathbf{a} \rangle$, and $c' \in P_{\omega\alpha+N+1} - \langle \mathbf{a} \rangle$.

Then *not* $\mathbf{a},c \leq_{2\alpha+1} \mathbf{a},c'$, but $\mathbf{a},c' \leq_{2\alpha+1} \mathbf{a},c$, so given **b** we can find **b'** so that $\mathbf{a},c,\mathbf{b} \leq_{2\alpha} \mathbf{a},c',\mathbf{b}'$.

(b) $G_{\omega\alpha+N+2}$ is infinite. ~~$U_{\omega\alpha+k}(G) \neq 0$ for some $k \geq N+2$.~~

Given $\bar{a}$, Choose $f \in P_{\omega\alpha+N+2} - \langle \bar{a} \rangle$ and $d_1 \in G_{\omega\alpha+N+1}$ so that $pd_1 = f$.

Let $p^N c_1 = d_1$ where $h(c_1) \geq \omega\alpha+1$.

~~Given a,~~ choose $d_2 \in P_{\omega\alpha+N} - P_{\omega\alpha+N+1} - \langle \mathbf{a}, c_1 \rangle$, proper with respect to $\langle \mathbf{a}, c_1 \rangle$, and choose c_2 with $p^N c_2 = d_2$ and $h(c_2) = \omega\alpha$.

Then c_2 is also proper with respect to $\langle \mathbf{a}, c_1 \rangle$.

Also take $c_3 = pc_2$. Then $h(c_3) = \omega\alpha+1$.

Let $c = c_1 + c_2$. Then $h(c) = \omega\alpha$ and $p^N c = d_1 + d_2$.

Note that $p(d_1+d_2) = f$ and $h(d_1+d_2) = \omega\alpha+N$.

Let $c' = c_1 + c_3$. Then $h(c') \geq \omega\alpha+1$ and $p^N c' = d_1 + pd_2 = d_1$.

Note that $pd_1 = f$ and $h(d_1) \geq \omega\alpha+N+1$.

Then *not* $\mathbf{a}, c \leq_{2\alpha+1} \mathbf{a}, c'$ but $\mathbf{a}, d_1, c' \leq_{2\alpha+1} \mathbf{a}, d_1+d_2, c$.

So, given $\mathbf{b}$, we can find $\mathbf{b}'$ so that $\mathbf{a}, c, \mathbf{b} \leq_{2\alpha} \mathbf{a}, c', \mathbf{b}'$.

Case 5. Let $\mathbf{p} = G_{\omega\alpha}$. Then

$$\{ \psi_{\mathbf{a}} : \mathbf{a} \textit{ is a finite sequence of elements from } G \}$$

is a $\Sigma^0_{2\alpha}$ Scott family for G, where *'x has height h_a'* means

$$\begin{cases} h(x) = h_a & \text{for } h_a < \omega\alpha \\ x = p \ \text{ for the } p = a \in \mathbf{p} & \text{for } h_a \geq \omega\alpha \end{cases}$$

However G is not $\hat{\Delta}^0_\alpha$ categorical, that is to say, not Δ^0_β categorical for any $\beta < \alpha$. This is clear because if it were we could use its Scott family as a Scott family for a group of length $\omega\beta+\omega$, which we have shown above to be not $\Delta^0_{2\beta+1}$ categorical, and so not Δ^0_β categorical. □

§5.3 Δ^0_α Stability.

A related notion is that of Δ^0_α stability. A recursive structure $\mathfrak{A}$ is said to be Δ^0_α *stable* if, for every recursive structure $\mathfrak{B} \cong \mathfrak{A}$, *every* isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ is Δ^0_α.

Clearly any structure with $2^{\aleph_0}$ automorphisms is not Δ^0_α stable for any α. As observed in [Ku]:

Proposition 5.3.1: A structure $\mathfrak{A}$ has $2^{\aleph_0}$ automorphisms iff for every finite sequence, $\mathbf{a}$, of elements from $\mathfrak{A}$, $(\mathfrak{A},\mathbf{a})$ is not *rigid.* That is to say, there is a non-trivial automorphism of $\mathfrak{A}$ which fixes $\mathbf{a}$.

Proof: Suppose $(\mathfrak{A},\mathbf{a})$ is rigid. There are only countably many one-one maps $f\colon \mathbf{a} \longrightarrow \mathbf{b}$ and each extends to at most one automorphism.
Conversely, suppose that for every finite sequence $\mathbf{a}$, $(\mathfrak{A},\mathbf{a})$ is not rigid. Then we may construct an infinite binary tree with a finite map at each node so that along each path in the tree the maps extend to a distinct automorphism of $\mathfrak{A}$.

At level 0 in the tree start with the empty map.

At each node of level $2n$, we have a map $\mathbf{a} \longmapsto \mathbf{b}$ which extends to an automorphism f. Let c be the next element in a listing of A not included in the domain of f. Since $(\mathfrak{A},\mathbf{b},f(c))$ is not rigid, there are two distinct automorphisms fixing $\mathbf{b}$ and $f(c)$. Suppose they map some element d to d_1 and d_2 respectively. Then at the successor nodes at level $2n+1$ put the maps

$$\mathbf{a},c,f^{-1}(d) \longmapsto \mathbf{b},f(c),d_1 \text{ and } \mathbf{a},c,f^{-1}(d) \longmapsto \mathbf{b},f(c),d_2.$$

At each node of level $2n+1$, we have a map $\mathbf{a} \longmapsto \mathbf{b}$ which extends to an automorphism f. Let c be the next element in a listing of B not included in the range of f. Since $(\mathfrak{A},\mathbf{b},c)$ is not rigid, there are two distinct automorphisms fixing $\mathbf{b}$ and c. Suppose they map d to d_1 and d_2 respectively. Then at the successor nodes at level $2n+2$ put the maps

$$\mathbf{a},f^{-1}(c),f^{-1}(d) \longmapsto \mathbf{b},c,d_1 \text{ and } \mathbf{a},f^{-1}(c),f^{-1}(d) \longmapsto \mathbf{b},c,d_2 \qquad \square$$

In the case of an infinite reduced abelian p-group G (with $p > 2$), $(G,\mathbf{a})$ is never rigid for any $\mathbf{a}$. [Choose $c \in P - \langle\mathbf{a}\rangle$. Then $\mathbf{a},c \longmapsto \mathbf{a},-c$ is a height preserving map and so, by the proof of Ulm's Theorem, extends to a non-trivial automorphism of G.]

In the case $p = 2$, $(G,\mathbf{a})$ is still never rigid, but we need to be a little more subtle in our argument. If $U_\alpha(G) = \aleph_0$ for some α, then by Lemma 4.2.1 we can find $c_1 \in P_\alpha$ proper with respect to $\langle\mathbf{a}\rangle$ and $c_2 \in P_\alpha$ proper with respect to $\langle\mathbf{a},c_1\rangle$ such that $h(c_1) = h(c_2) = \alpha$. Then $\mathbf{a},c_1 \longmapsto \mathbf{a},c_2$ is a height preserving map.

Otherwise there are infinitely many α for which $U_\alpha(G) \geq 1$. Given $\mathbf{a}$, choose α and $\beta > \alpha$ so that $\langle\mathbf{a}\rangle$ contains no elements of height α or β, and $U_\alpha(G)$, $U_\beta(G) \geq 1$. Then by Lemma 4.2.1, we can find c proper with respect to $\langle\mathbf{a}\rangle$ and c' proper with respect to $\langle\mathbf{a},c\rangle$ so that $h(c) = \alpha$, $h(c') = \beta$ and $pc' = 0$. Then $\mathbf{a},c \longmapsto \mathbf{a},c+c'$ is a height preserving map.

Thus we have the following:

Proposition 5.3.2: No infinite reduced abelian p-group is Δ^0_α stable for any α. □

§5.4 Pairs of Structures.

In [AKn] conditions are given for the possibility of the construction, uniformly in n, of recursive structures $\mathfrak{C}_n$ isomorphic to a given structure $\mathfrak{A}$ if $n \in S$ and to another given structure $\mathfrak{B}$ if $n \notin S$.

We take the notation

$$\begin{cases} \mathfrak{A} \text{ if } \Pi^0_\alpha \\ \mathfrak{B} \text{ if not} \end{cases}$$

to mean "For every Π^0_α set S there are structures $\mathfrak{C}_n$, recursive uniformly in n, such that

$$\mathfrak{C}_n \cong \begin{cases} \mathfrak{A} \text{ if } n \in S \\ \mathfrak{B} \text{ if } n \notin S \end{cases} ."$$

The construction in [AKn] uses an α-system, so the back and forth relations $\leq_\beta$ are needed. However, since we are dealing with several different structures in the one construction, we need the following generalization.

Definition 5.4.1: A *recursive family* of recursive structures is an indexed family $\{ \mathfrak{A}_i : i \in I \}$ for which I is a recursive set and each $\mathfrak{A}_i$ is a recursive structure, uniformly in i.

Such a recursive family is *α-friendly* (where $\alpha < \omega_1^{CK}$) if, using some notation for α, the relation $\Pi_\beta(\mathfrak{A}_i,\mathbf{a}) \subseteq \Pi_\beta(\mathfrak{A}_j,\mathbf{b})$ is r.e. between $i,j \in I$, $\beta < \alpha$ and sequences $\mathbf{a}$ from $\mathfrak{A}_i$ and $\mathbf{b}$ from $\mathfrak{A}_j$.

We will apply the following result to reduced abelian p-groups.

Proposition 5.4.2: Let $\mathfrak{A}$ and $\mathfrak{B}$ be recursive structures for which $\{\mathfrak{A},\mathfrak{B}\}$ is an α-friendly family. Then

(i) if $\Pi_\alpha(\mathfrak{B}) \subseteq \Pi_\alpha(\mathfrak{A})$, then $\begin{cases} \mathfrak{A} \text{ if } \Pi^0_\alpha \\ \mathfrak{B} \text{ if not} \end{cases}$.

(ii) if $\Pi_\alpha(\mathfrak{B}) = \Pi_\alpha(\mathfrak{A})$, then $\begin{cases} \mathfrak{A} \text{ if } \Delta^0_{\alpha+1} \\ \mathfrak{B} \text{ if not} \end{cases}$.

Proof: See [AKn]. □

In order to do so, we need some families of α-friendly p-groups.

Let G^β be a recursive copy with recursive height function of the reduced abelian p-group with length β and Ulm invariants $U_\gamma(G^\beta) = \aleph_0$ for all $\gamma < \beta$.

Let $H^{\omega\beta}$ be a recursive copy with recursive height function of the reduced abelian p-group with length $\omega\beta$ and Ulm invariants $U_\gamma(G^{\omega\beta}) = 1$ for all $\gamma < \omega\beta$.

Proposition 5.4.3: The families $\{ G^\beta : \beta < \alpha' \}$ and $\{ H^{\omega\beta} : \beta < \alpha' \}$ are α-friendly for any $\alpha,\alpha' < \omega_1^{CK}$.

Proof: For $\mathfrak{A},\mathfrak{B}$ both from one or the other family we have:

(i) $\Pi_{2\alpha}(\mathfrak{A},\mathbf{a}) \leq \Pi_{2\alpha}(\mathfrak{B},\mathbf{b})$ iff

(a) $\lambda(\mathfrak{A}) = \lambda(\mathfrak{B}) \leq \omega\alpha$ or $\lambda(\mathfrak{A}),\lambda(\mathfrak{B}) > \omega\alpha$; and

(b) the map $f : \mathbf{b} \longrightarrow \mathbf{a}$ which maps corresponding members of the sequences onto each other extends to an isomorphism $f : \langle\mathbf{b}\rangle \cong \langle\mathbf{a}\rangle$; and

(c) for every $b \in \langle\mathbf{b}\rangle$ and $a = f(b)$ we have

$$h(b) = h(a) < \omega\alpha \quad \textit{or} \quad h(b),h(a) \geq \omega\alpha.$$

(ii) $\Pi_{2\alpha+1}(\mathfrak{A},\mathbf{a}) \leq \Pi_{2\alpha+1}(\mathfrak{B},\mathbf{b})$ iff

(a) $\lambda(\mathfrak{A}) = \lambda(\mathfrak{B}) \leq \omega\alpha$ *or*

$\lambda(\mathfrak{B}) > \omega\alpha$ and $\lambda(\mathfrak{A}) \geq \min \{ \lambda(\mathfrak{B}), \omega\alpha+\omega \}$; and

(b) the map $f : \mathbf{b} \longrightarrow \mathbf{a}$ which maps corresponding members of the sequences onto each other extends to an isomorphism $f : \langle\mathbf{b}\rangle \cong \langle\mathbf{a}\rangle$; and

(c) for every $b \in \langle\mathbf{b}\rangle$ and $a = f(b)$ we have

$$h(b) = h(a) < \omega\alpha, \text{ or}$$

$$h(b) \geq \omega\alpha \text{ and } h(a) \geq \min \{h(b), \omega\alpha+\omega\}.$$

This is proved in essentially the same way as Proposition 4.2.2. The conditions on the length of the groups ensure that appropriate elements are available to be mapped to, in the same way as the conditions on the heights of already mapped elements. Note that the structure of the groups in the two families considered ensures that for any given height less than the length of the group there are infinitely many elements of at least that height. Thus the construction in the proof can never run out of elements. □

We may now obtain the following results.

Proposition 5.4.4: For any $\alpha, \gamma < \omega_1^{CK}$

(i) $\begin{cases} G^{\omega\alpha+1} & \text{if } \Pi^0_{2\alpha+1} \\ G^{\omega\alpha+1+\gamma} & \text{if not} \end{cases}$

(ii) $\begin{cases} H^{\omega(\alpha+1)} & \text{if } \Pi^0_{2\alpha+1} \\ H^{\omega(\alpha+1+\gamma)} & \text{if not} \end{cases}$

Proof: These come from (ii) in the proof of Proposition 5.4.3 applied to the empty sequences **a** and **b**, and Proposition 5.4.2 (i).
□

CHAPTER 6. A PARTIAL ORDERING.

The examples we have considered up to now have been of recursive structures because every r.e. p-group or linear ordering is already a recursive structure. In this chapter we consider a simple example which shows a difference in its behaviour when considered as an r.e. structure and its behaviour when considered as a recursive structure.

An r.e. linear ordering is a recursive structure because the order relation is total. So, for $x \neq y$, *not* $x < y$ iff $y < x$. If the order relation is only partial this argument does not work. Thus we may have r.e. partial orderings that are not recursive.

Let P be a recursive copy of the shuffle sum ([Rs], defn 7.14) of the orderings $1,2,\ldots,\omega$ with linear order relation $\angle$ and recursive distance function $d : P^2 \longrightarrow \omega \cup \{\infty\}$. Define a partial order on P by

$$x < y \text{ iff } x \angle y \;\&\; d(x,y) = \infty.$$

(So we have a 'linear shuffle sum' of incomparable sets of elements of size $1,2,\ldots,\omega$.)

Then $\mathfrak{P} = (P,<)$ can be considered as a recursive structure or as an r.e. structure.

Let R be the set of elements of P in some infinite anti-chain. Then in $\mathfrak{P}$, $x \in R$ iff x is in arbitrarily large finite antichains:

$$x \in R \longleftrightarrow \bigwedge\!\!\!\bigwedge_{k \in \mathbb{N}} \exists y \,(\, y_k = x \;\&\; \bigwedge_{1 \le i \neq j \le k} not\ y_i \le y_j \,)$$

So R is formally Π^0_2 and formally Π^{0+}_3. (*not* $y_i \le y_j$ is a basic negative formula, so $\exists \mathbf{y}\ (\ y_k = x\ \ \&\ \bigwedge_{1\le i\ne j\le k}$ *not* $y_i \le y_j\ )$ is Σ^+_2, rather than Σ^+_1.)

We will show that R is not formally Σ^0_2 and not formally Σ^{0+}_3.

Write $\mathbf{x} \equiv_0 \mathbf{y}$ if corresponding elements of $\mathbf{x}$ and $\mathbf{y}$ stand in exactly the same order relations. Let $\sigma(x)$ be the number of elements incomparable with x.

Then we claim that $\mathbf{a} \le_1 \mathbf{b}$ iff $\mathbf{a} \equiv_0 \mathbf{b}$ and $\sigma(a_i) \ge \sigma(b_i)$ for each i.

If the condition is satisfied we will show how, given $\mathbf{d}$, we can find appropriate $\mathbf{c}$ to match $\mathbf{d}$ with. By the density of the shuffle sum, any ds comparable to all the $\mathbf{b}$ can be matched with appropriate cs in the correct order relations to the $\mathbf{a}$. Any ds incomparable with some b can be matched because there are enough elements incomparable with the corresponding a.

On the other hand, if the condition is not satisfied, we can choose more elements incomparable with some b than can be matched with elements incomparable to the corresponding a, thus causing a violation of the correspondence in order relations.

Also $\mathbf{a} \le^+_1 \mathbf{b}$ iff for every $\mathbf{d}$ and every basic positive formula φ with $\mathfrak{P} \models \varphi(\mathbf{b},\mathbf{d})$ there is $\mathbf{c}$ such that $\mathfrak{P} \models \varphi(\mathbf{a},\mathbf{c})$.

Now a basic positive formula is essentially a conjunction of positive statements of order relations. So, by the density of the shuffle sum, we can always find such $\mathbf{c}$ provided that whenever

$b_i < b_j$, or $b_i = b_j$, or $b_i > b_j$, we have the corresponding $a_i < a_j$, or $a_i = a_j$, or $a_i > a_j$.

Finally, $\mathbf{a} \leq_2^+ \mathbf{b}$ iff for every $\mathbf{d}$ and every basic positive formula φ with $\mathfrak{P} \models \varphi(\mathbf{b},\mathbf{d})$ there is $\mathbf{c}$ such that $\mathbf{a},\mathbf{c} \geq_1^+ \mathbf{b},\mathbf{d}$ and $\mathfrak{P} \models \varphi(\mathbf{a},\mathbf{c})$.

Suppose that $\mathbf{a} \leq_1 \mathbf{b}$. Then for every $\mathbf{d}$ there is $\mathbf{c}$ such that $\mathbf{a},\mathbf{c} \equiv_0 \mathbf{b},\mathbf{d}$. This implies $\mathbf{a} \leq_2^+ \mathbf{b}$. Also, by Corollary 2.3.6, $\mathbf{a} \leq_2^+ \mathbf{b}$ implies $\mathbf{a} \leq_1 \mathbf{b}$.

So $\mathbf{a} \leq_2^+ \mathbf{b}$ iff $\mathbf{a} \leq_1 \mathbf{b}$.

Thus the following argument shows that R is both not formally Σ_2^0 and not formally Σ_3^{0+}.

Given $\mathbf{a}$, choose $r \in R$ so that r is comparable with each $a \in \mathbf{a}$. Then, given $\mathbf{b}$, determine how many elements of $\mathbf{b}$ are incomparable with r. Find a finite maximal set of incomparable elements with more than this many elements, in the same order relations with all the rest of $\mathbf{a},\mathbf{b}$, and map r and those bs incomparable with r into this set. This shows that

$$r \in R - Rcl_2(\mathbf{a}) = R - Rcl_3^+(\mathbf{a}).$$

So, by Theorem 2.7.1, there is a recursive structure isomorphic to $\mathfrak{P}$ in which $\{ x : \sigma(x) = \infty \}$ is not a Σ_2^0 set, and this is the best possible result as R is formally Π_2^0. By contrast however, by Theorem 2.6.1, there is an r.e. structure isomorphic to $\mathfrak{P}$ in which $\{ x : \sigma(x) = \infty \}$ is not a Σ_3^0 set.

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INTRINSICALLY Σ^0_α RELATIONS

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A recursive structure $\mathfrak{A}$ consists of a recursive underlying set $|\mathfrak{A}|$, and recursive relations and operations on $|\mathfrak{A}|$. A new relation on $\mathfrak{A}$ is called *formally* Σ^0_α if it has a definition as a certain type of infinitary formula involving the given relations on the structure. It is called *intrinsically* Σ^0_α if the corresponding relation in any isomorphic recursive structure forms a Σ^0_α set (or set of ordered tuples).

In [3] it is shown that given a condition guaranteeing a certain amount of extra decidability in the structure these two notions coincide for the Σ^0_1 (recursively enumerable) case. Here we deal with the analogous result for the general case, for α a constructive ordinal. We give a direct (infinite injury) construction for the case $\alpha = 2$, together with several examples, which demonstrate that the decidability conditions required are satisfiable in natural examples. Then, applying a theorem of C.J. Ash [1, 2], we deal with the general case.

0. Introduction

A recursive structure may be thought of as a classical mathematical object Gödel-numbered in an effective way. Our interest here is in the scope for variation in the effective properties of the structure under different Gödel-numberings. Thus we consider recursive structures classically isomorphic to a given recursive structure.

In a recursive linear ordering the order relation must be recursive, by definition. Here is an example of what possibilities there are for other relations.

Proposition 1. *In a recursive well-ordering of type ω^2, the set of left limit points*

$$L = \{x \mid \forall y\, \exists z\, [y < z < x \vee z < x \leq y]\}$$

is Π^0_2, and its complement L' has an infinite recursive subset.

Conversely, given any infinite Π^0_2 set whose complement has an infinite recursive subset, there is a recursive well-ordering of type ω^2, for which this set is the set of left limit points.

Proof. By the definition of L and the fact that $<$ is recursive, we see that L is Π^0_2. Furthermore, if a is the first limit point in the ordering, then $\{x \mid x < a\}$ is an infinite recursive subset of L'.

For the converse, suppose we are given such a set, P. That is, we are given recursive relations T and R such that

$$x \in P \iff \forall y \, \exists z \, T(x, y, z)$$

and R is an infinite subset of P'

We will construct, by stages, a well-ordering of type ω^2. It will be recursive because the construction will simply involve placing elements in the ordering at various stages, and once they are placed they will not subsequently be moved. Triplets will be systematically tested for membership of T, one per stage.

Stage s. (i) Place the $(s+1)$-th element of R' after all the elements of the ordering placed prior to stage s.

(ii) For the first s elements of R', $a_1, \ldots, a_s$, if $T(a_i, k, l)$ appears by stage s for some l and there are exactly k elements of R immediately before a_i, then place the next unused element of R immediately before a_i.

Note that this construction is effective because R is recursive, and at each stage there is only a finite amount of checking to be done.

The construction eventually places all the elements of R' in an ordering of type ω and fills in the gaps between them (with the elements of R) according as they are in P or not.

If a_i is in P, then for each k, at some stage after there are k elements immediately before a_i, a $(k+1)$-th element will be placed there. Thus, an ordering of type ω will be built up immediately before a_i, so a_i will be a left limit point.

If, however, $a_i \notin P$, then for some k, there is no l with $T(a_i, k, l)$, so a_i will have no more than k elements of R immediately before it. That is, a_i will be a successor element. Since we assume that P is infinite, the construction produces a well-ordering of type ω^2, as required. $\square$

Note that, in particular, P can be chosen to be not Σ_2^0. So the set of all left limit points in a recursive well-ordering of type ω^2 is not intrinsically Σ_2^0.

However, L has a definition as a formally Π_2^0 set, and so as we will see it is intrinsically Π_2^0.

1. Infinitary formulae

Strictly speaking we are working within Kleene's Constructive Ordinals (see, for example, [5]) and to speak of an ordinal α is really to speak of a notation a for α, but we will usually suppress this detail. A more formal definition is given in [2].

The (*recursive*) Σ_α and Π_α *formulae* are defined recursively: The Σ_0 and Π_0 formulae are the finitary quantifier-free formulae. The $\Sigma_{\alpha+1}$ formulae are those of the form $\bigvee\!\!\!\bigvee_{n \in W_e} \exists \boldsymbol{y}_n \, \theta_n$, where the θ_n are Π_α formulae indexed by their Gödel numbers, and the $\exists \boldsymbol{y}_n$ are finite blocks of existential quantifiers. That is, they are r.e. disjunctions of $\exists\, \Pi_\alpha$ formulae. Similarly, the $\Pi_{\alpha+1}$ formulae are r.e. conjunctions of $\forall\, \Sigma_\alpha$ formulae. For α a limit ordinal, the Σ_α formulae are those of the form $\bigvee\!\!\!\bigvee_{n\in W_e} \theta_n$ where θ_n is a Σ_{α_n} formula, and $\{\alpha_n\}$ is the sequence of ordinals having limit α that is given by the notation for α. Similarly the Π_α formulae are those of the form $\bigwedge\!\!\!\bigwedge_{n\in W_e} \theta_n$ where θ_n is a Π_{α_n} formula.

Definition. A relation R on a recursive structure is *formally* Σ^0_α if it has a definition

$$\mathfrak{A} \vDash \boldsymbol{x} \in R \leftrightarrow \varphi(\boldsymbol{p}, \boldsymbol{x})$$

where φ is a Σ_α formula involving only the given (i.e. recursive, by definition) relations of $\mathfrak{A}$ and $\boldsymbol{p}$ is a finite list of parameters from $|\mathfrak{A}|$.

2. Intrinsically Σ^0_α relations

Definition. A relation R on a recursive structure is *intrinsically* Σ^0_α if for every isomorphism between $\mathfrak{A}$ and another recursive structure $f:\mathfrak{A}\cong\mathfrak{B}$, $f(R)$ forms a Σ^0_α set.

(Throughout this paper we will think of relations as unary. Other relations can be considered in a suitable product structure which turns them into unary relations.)

The following lemma is the 'easy direction' of this paper.

Lemma 1. *If a relation R is formally Σ^0_α on a recursive structure $\mathfrak{A}$, then it is intrinsically Σ^0_α on $\mathfrak{A}$.*

Proof. Suppose $\mathfrak{A} \vDash x \in R \leftrightarrow \varphi(\boldsymbol{p}, x)$ where φ is Σ_α, and $f:\mathfrak{A}\cong\mathfrak{B}$ where $\mathfrak{B}$ is another recursive structure. Then $\mathfrak{B} \vDash y \in f(R) \leftrightarrow \varphi'(f(\boldsymbol{p}), y)$, where φ' is obtained from φ by replacing the recursive relations of $\mathfrak{A}$ by the corresponding relations of $\mathfrak{B}$, and $f(\boldsymbol{p})$ means $f(p_1), \ldots, f(p_n)$ for $\boldsymbol{p} = p_1, \ldots, p_n$. So $f(R)$ has a formally Σ^0_α definition in $\mathfrak{B}$, and it is easy to show that Σ_α formulae uniformly represent Σ^0_α sets.

3. The case $\alpha = 2$

We wish to establish a partial converse to Lemma 1: we will need to assume some extra decidability in the structures considered. (In [4] it is shown, for the case $\alpha = 1$, that such conditions cannot be entirely dispensed with.)

First we deal with the case $\alpha = 2$ separately, as the direct argument allows a slightly more general result, which is noted at the end of the proof.

Definition. For $\boldsymbol{a}, \boldsymbol{b} \in |\mathfrak{A}|$, we write $\boldsymbol{a} \leqslant_1 \boldsymbol{b}$ if $\boldsymbol{b}$ satisfies the universal type of $\boldsymbol{a}$. That is, if every universal formula true of $\boldsymbol{a}$ is also true of $\boldsymbol{b}$.

Theorem 1. *Let R be a relation on a recursive structure $\mathfrak{A}$ satisfying conditions* (A), (B) *and* (C). *Then R is intrinsically Σ_2^0 iff it is formally Σ_2^0.*

Conditions. (A) *The existential diagram of $\mathfrak{A}$ is recursive, and R is recursive.*

(B) *There is a recursive procedure to determine, for $\boldsymbol{a}, \boldsymbol{a}' \in |\mathfrak{A}|$, whether $\boldsymbol{a} \leqslant_1 \boldsymbol{a}'$.*

(C) *There is a recursive procedure to determine for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in |\mathfrak{A}|$ with $\boldsymbol{a} \in R$, whether there exist $\boldsymbol{a}', \boldsymbol{b}' \in |\mathfrak{A}|$ with $\boldsymbol{a}' \notin R$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \leqslant_1 \boldsymbol{a}', \boldsymbol{b}, \boldsymbol{c}$.*

Proof. We will take R to be a unary relation. If R is n-ary, it can be considered as unary on a suitably defined product structure of n-tuples from $|\mathfrak{A}|$.

Given $\mathfrak{A}$ and R satisfying (A), (B) and (C) with R not formally Σ_2^0, we want to construct a recursive $\mathfrak{B}$ and a classical isomorphism $f: \mathfrak{A} \cong \mathfrak{B}$ such that $f(R)$ is not a Σ_2^0 set. To this end we need witnesses x_e to ensure that $f(R) \neq S_e$, the e-th Σ_2^0 set.

We may suppose that we have uniformly recursive total functions $\mu_e: \mathbb{N}^2 \to 2$ with $y \in S_e \Leftrightarrow \neg(\mathbf{U}s)[\mu_e(s, y) = 0]$. That is, there are not infinitely many stages s with $\mu_e(s, y) = 0$.

We want to arrange that $f(x_e) \in S_e \Leftrightarrow x_e \notin R$. To attempt to do this we will choose a suitable witness, $x \in R$, and a tentative image, $f(x) = y$, for it. If $\mu_e(x, y) = 0$ for infinitely many s, then $y \notin S_e$, and this is exactly what we need. If, however, $\mu_e(x, y) = 0$ only finitely often, then we will (because of the way we have chosen x) be able to choose some $x' \notin R$ to map to y instead of x.

In more detail this works as follows: Choose a flexible $x \in R$ and define $f(x)$. Also define f' so that $f'^{-1} \circ f(x) \notin R$. So long as $\mu_e(s, f(x)) = 1$, we are happy with f' as a candidate for extending to the required isomorphism. If at some s, however, $\mu_e(s, f(x)) = 0$, then we discard f' and return to f and extend it a bit. Now we find a new f' (the old one is not necessarily compatible with what we have added to f), and work with it.

So every time $\mu_e(s, f(x)) = 0$ we add a bit to our construction of f. If this happens infinitely often, then we will have had time to completely define f, and it can be used as our isomorphism. If, on the other hand, after some stage $\mu_e(s, f(x))$ is always 1, then we are left with a suitable f' which has been extended infinitely often, and it can be used for the isomorphism.

That is an outline of the procedure for dealing with a single requirement. We fit these together by forbidding an attempt to satisfy a requirement to upset a lower numbered (that is, higher priority) requirement. This leads us to cast the construction in the form of a tree of functions.

Let $A = |\mathfrak{A}|$, and take an infinite recursive set B to be used as $|\mathfrak{B}|$.

At each stage we will have irrevocably decided finitely many sentences of the atomic diagram of $\mathfrak{B}$. (This is to ensure that $\mathfrak{B}$ is a recursive structure.) Let the finite subset of the atomic diagram of $\mathfrak{B}$ defined by stage s be $\mathfrak{B}^s$. For simplicity we will assume that $\mathfrak{B}^s$ is closed under finite conjunctions and that if b_i, b_j $(i \neq j)$ appear in $\mathfrak{B}^s$, then $b_i \neq b_j$ is also in $\mathfrak{B}^s$.

We define a finite 1–1 partial function $f: A \to B$ to be *coherent* w.r.t. $\mathfrak{B}^s$ if whenever $\phi(f(\boldsymbol{a}), \boldsymbol{b}) \in \mathfrak{B}^s$ then $\mathfrak{A} \models \exists \boldsymbol{y}\, \phi(\boldsymbol{a}, \boldsymbol{y})$.

This is equivalent to the condition that f extends to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ for some total $\mathfrak{B} \supseteq \mathfrak{B}^s$.

Also, as $\mathfrak{A}$ is assumed to have a recursive existential diagram, given finite $A_0 \subseteq A$, $B_0 \subseteq B$, f can be effectively extended to a coherent map having A_0 in its domain and B_0 in its range.

Let T be the full binary tree of finite sequences of 0's and 1's. The root node of T is thus the empty sequence, $\emptyset$.

At stage s we will have defined for finitely many nodes $p \in T$ a finite 1–1 partial map $f^s_P: A \to B$ which is coherent w.r.t. $\mathfrak{B}^s$. We say a node is defined if there is a function defined at that node.

Construction. At stage 0 all nodes are undefined and $\mathfrak{B}^0$ is empty.

$f^s_\emptyset = \emptyset$ at every stage s. At the start of each stage s, set $f^s_p = f^{s-1}_p$ for those nodes p at which f^{s-1}_p is defined.

At stage s, starting at the root node, we will step down the tree choosing our path according to the current information available about the S_e. Eventually we will come to an undefined node and here we will define a new partial function, thus extending the tree. We give the procedure inductively:

Assume we are at node p, where f^s_p is defined. Let $e = \text{length}(p)$ (which is the level of node p in T), and let q be the bottom-right most node that is presently defined. (So q is the largest defined node in lexicographic ordering, and in fact q will always be p or a node below p, because of the way our construction works.)

Define x^s_p as follows:

Let $\boldsymbol{a} = \text{dom}(f^s_p)$. Let $\boldsymbol{b}$ consist of $\text{dom}(f^s_q)$, together with elements that are asserted to exist by the condition of coherence for f^s_q w.r.t. $\mathscr{B}^s$, leaving out those elements already in $\boldsymbol{a}$.

Let $\Phi(x, \boldsymbol{y}, \boldsymbol{z})$ be the recursive infinitary Π_1 formula defining the universal type of $x, \boldsymbol{b}, \boldsymbol{a}$, namely the conjunction of all universal formulae satisfied by $x, \boldsymbol{b}, \boldsymbol{a}$. This conjunction is recursive because the existential diagram of $\mathfrak{A}$ is recursive.

Then we take x^s_p to be the first element, x, of R such that there are $x' \notin R$ and $\boldsymbol{b}'$ with $\mathfrak{A} \models \Phi(x', \boldsymbol{b}, \boldsymbol{a})$.

Having f^s_p and x^s_p, we now ask whether f^s_{p0} is defined and $x^s_p \in \text{dom}(f^s_{p0})$.

(i) If not, we have travelled as far down the tree as was previously defined or else x^s_p has changed to a now inconvenient element. Call the node we are at the *current* node p^s.

We define f^s_{p0} and proceed to the next stage, throwing away any other work on nodes below p. (This will cause no problems, because on nodes that are eventually important in the construction x^s_p will settle down.)

So undefine all nodes below p. Define f^s_{p0} to extend f^s_p, have a_e, $x^s_p \in \mathrm{dom}(f^s_{p0})$; $b_e \in \mathrm{ran}(f^s_{p0})$ and be coherent w.r.t. $\mathscr{B}^s$. Add to $\mathfrak{B}^s$ the first undecided atomic formula or its negation so as to keep f^s_{p0} coherent, thus forming $\mathscr{B}^{s+1}$. Now go back to $\emptyset$ and start stage $s+1$.

(ii) If $f^s_{p0}(x^s_p)$ is defined, then calculate $\mu_e(s, f^s_{p0}(x^s_p))$.

(a) If this is 0, then undefine node $p1$ and all nodes below it, and go to node $p0$.

(b) If $\mu_e(s, f^s_{p0}(x^s_p)) = 1$, then we branch to the right.

If f^s_{p1} is not defined, then define it to extend f^s_p, have $a_e \in \mathrm{dom}(f^s_{p1})$; $\mathrm{ran}(f^s_q) \subseteq \mathrm{ran}(f^s_{p1})$; $(f^s_{p1})^{-1} \circ f^s_q(x^s_p) \notin R$; be coherent w.r.t. $\mathscr{B}^s$; and be such that $(f^s_{p1})^{-1} \circ f^s_q$ preserves universal formulae.

(There are a lot of conditions here, but in fact x^s_p has been chosen just so this can be done.)

Now go to node $p1$. (If we have not just defined f^s_{p1}, then we continue stepping down the tree. If we have just defined it then we will now be in case (i).) □

We must now check that the construction is effective, and that all partial functions remain coherent.

For each s and p, x^s_p must exist because otherwise R is formally Σ^0_2: If every $x \in R$ is constrained to be in R by some Π_1 formula, Φ_x, then

$$r \in R \leftrightarrow \bigvee\!\!\!\bigvee_{x \in R} \exists \boldsymbol{y}\ \Phi_x(r, \boldsymbol{y}, \boldsymbol{a}).$$

Furthermore, x^s_p can be found effectively by condition (C).

In case (i) we can define f^s_{p0}, effectively because of the coherence of f^s_p, as noted above.

In case (ii)(b), we know that such a function exists by the choice of x^s_p. It can be found effectively by using condition (B).

Lemma 2. *Every f^s_p remains coherent.*

Proof. Define the relation $f \lhd g$ to mean that $\mathrm{ran}(g) \supseteq \mathrm{ran}(f)$ and $g^{-1} \circ f$ preserves universal formulae.

We now observe that if p is to the right of or below q (i.e. $q \leqslant p$ in lexicographical ordering), then we have $f^s_p \lhd f^s_q$. This is because $\lhd$ is transitive, and whenever we define an f^s_p we ensure $f^s_q \lhd f^s_p$ where q is the bottom right-most previously defined node. All previously defined functions to the right of node p are discarded in the process of stepping through the tree to p.

So it remains to show that if g is coherent and $f \lhd g$, then f is coherent.

Suppose that $\mathfrak{B}^s \models \phi[f(\boldsymbol{a}), \boldsymbol{b}]$. Then we want to show that $\mathfrak{A} \models \exists \boldsymbol{y}\ \phi(\boldsymbol{a}, \boldsymbol{y})$.

Now $f = g \circ (g^{-1} \circ f)$, so

$$\mathfrak{B}^s \models \phi[g \circ (g^{-1} \circ f)(\boldsymbol{a}), \boldsymbol{b}].$$

Since g is assumed coherent, we have that

$$\mathfrak{A} \models \exists \boldsymbol{y}\, \phi[(g^{-1} \circ f)(\boldsymbol{a}), \boldsymbol{y}].$$

Now because $g^{-1} \circ f$ preserves universal formula, if $\mathfrak{A} \models \forall \boldsymbol{y} \neg\phi(\boldsymbol{a}, \boldsymbol{y})$, then

$$\mathfrak{A} \models \forall \boldsymbol{y} \neg\phi(g^{-1} \circ f(\boldsymbol{a}), \boldsymbol{y})).$$

This contradicts the above, so $\mathfrak{A} \models \exists \boldsymbol{y}\, \phi(\boldsymbol{a}, \boldsymbol{y})$, and therefore f is coherent. $\square$

Now for the desired model we choose an infinite branch $\{p_0, p_1, \ldots\} = P$ through T inductively as follows.

– p_0 is the root node.

– p_{n+1} is p_{n0} if p^s is below p_{n0} for infinitely many s, and otherwise p_{n+1} is p_{n1}.

It follows from this definition that if $p \in P$, then p^s is to the left of p only finitely often.

This is not effective, as the isomorphism that we will obtain is not Δ^0_2. However, we have ensured that the construction of $\mathfrak{B}$ is recursive, and it is enough that the isomorphism f can be shown to exist.

Lemma 3. *For each $p_n \in P$, there is a stage when x_{p_n} and f_{p_n} are defined and never again change.*

Proof. By induction on n. Since p_0 is the root node, f_{p_0} never changes after it is defined.

Once f_{p_n} has stopped changing, if x_{p_n} does not eventually stop changing, then the construction provides an r.e. list of formulae constraining each element of R to be in R, with parameters $\mathrm{dom}(f_{p_n})$. The disjunction of these formulae would be a formally Σ^0_2 definition of R.

So eventually f_{p_n} and x_{p_n} stop changing. But there are only finitely many p^s to the left of the p_{n+1}, and these stages s are the only stages at which $f_{p_{n+1}}$ can change. So eventually $f_{p_{n+1}}$ stops changing. $\square$

So for each $p \in P$, we have

$$f_p = \lim_s f^s_p \quad \text{and} \quad x_p = \lim_s x^s_p.$$

The f_p's form a chain of coherent functions with $a_e \in \mathrm{dom}(f_p)$ and $b_e \in \mathrm{ran}(f_p)$ where $e = \mathrm{length}(p)$. Hence $f = \bigcup_{p \in P} f_n$ is an isomorphism $f : \mathfrak{A} \cong \mathfrak{B}$.

We now check that f satisfies our requirements. Let $e = \mathrm{length}(p)$.

If $p0 \in P$, then infinitely often p^s is below $p0$. That is, $\mathbf{U}s\,[\mu_e(s, f^s_{p0}(x^s_p)) = 0]$. So $\mathbf{U}s\,[\mu_e(s, f_{p0}(x_p)) = 0]$, i.e., $f(x_p) = f_{p0}(x_p) \notin S_e$.

But $x_p \in R$. So take $x_e = x_p$ as our witness that $f(R) \neq S_e$.

If $p1 \in P$, then p^s is below $p0$ only finitely often. In this case f_{p0} also settles down since $p0$ is not visited after some stage.

We have $\neg\mathbf{U}s\ [\mu_e(s, f^s_{p0}(x^s_p)) = 0]$. So $\neg\mathbf{U}s\ [\mu_e(s, f_{p0}(x_p)) = 0]$. That is, $f_{p0}(x_p) \in S_e$.

But f_{p1} was chosen so that $f^{-1}_{p1} \circ f_{p0}(x_p) \notin R$. So take $x_e = f^{-1}_{p1} \circ f_{p0}(x_p)$. Then $f(x_e) = f_{p1}(x_e) = f_{p0}(x_p) \in S_e$ and $x_e \notin R$.

Thus we have, for every e, $x_e \in R \Leftrightarrow f(x_e) \notin S_e$.

Hence $f(R) \neq S_e$ for every e. So $f(R)$ is not Σ^0_2, and so R is not intrinsically Σ^0_2. $\square$

Condition (C) can be relaxed to the following:

(C′) There is a Δ^0_2 procedure to determine for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in |\mathfrak{A}|$ with $\boldsymbol{a} \in R$, whether there exist $\boldsymbol{a}', \boldsymbol{b}' \in |\mathfrak{A}|$ with $\boldsymbol{a}' \notin R$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \leqslant_1 \boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}$.

This is because we can think of this Δ^0_2 procedure as always giving an answer when questioned, but not necessarily giving the right answer. However, eventually at some stage and all subsequent stages, it does give the correct answer.

So when we apply (C′) to find a suitable x^s_p, if we get an answer of 'yes' for particular $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, we immediately start searching for suitable $\boldsymbol{a}', \boldsymbol{b}'$ as confirmation of this.

We know that either we will find such $\boldsymbol{a}', \boldsymbol{b}'$ (which can be checked effectively by condition (B) and the fact that R is recursive), in which case the answer is in fact true, or else (C′) will subsequently 'change its mind', and we will know to try another candidate for x^s_p.

The only other alteration required is that while searching for x^s_p, we must keep rechecking each candidate in case a 'no' answer subsequently changes to 'yes'.

So the construction remains effective.

4. Examples: linear orders

A recursive linear order $\mathfrak{A} = (A, <)$ consists of a recursive set A and a recursive order relation $<$ on A which is total.

We will consider limit points.

$L = \{\text{left limit points}\} = \{x \mid \forall y\ \exists z\ (y < z < x \text{ or } z < x \leqslant y)\}$,

$R = \{\text{right limits points}\} = \{x \mid \forall y\ \exists z\ (x < z < y \text{ or } y \leqslant x < z)\}$,

$T = \{\text{two-sided limit points}\} = L \cap R$.

We have seen that for $\mathfrak{A} = \omega^2$, L is not intrinsically Σ^0_2. If $\mathfrak{A} = k \cdot \eta$ (for some $k \in \mathbb{N}$), then we can define L by

$$L = \left\{x \mid \exists y_1 \cdots y_k\ \forall z \left[x = y_1 < y_2 < \cdots < y_k \ \&\ \bigwedge_{i=1,\ldots,k-1} \neg(y_i < z < y_{i+1})\right]\right\}.$$

So L is formally Σ^0_2, and so certainly intrinsically Σ^0_2. We will see that, in a sense, $k \cdot \eta$ is the only recursive linear order for which L is formally Σ^0_2.

(ω is the order type of the natural numbers, η is the order type of the rationals, and for any order type β, β^* is the reverse type.)

Define a distance function $d: A^2 \to \mathbb{N} \cup \{\infty\}$ by $d(x, y) =$ number of points strictly between x and y.

We will also use the notion of a block. We say

$$\text{block}(x, y) \overset{\text{def}}{\Leftrightarrow} d(x, y) < \infty.$$

$$\{y \mid \text{block}(x, y)\} \text{ is the block containing } x.$$

A block is a maximal segment of a linear order obtained by just adding successors and predecessors of elements already in the segment.

It will be convenient to add two new end points to each linear order we consider. Add $\mathbf{0}$ and $\mathbf{1}$ to A with the order relations

$$\mathbf{0} < x < \mathbf{1} \quad \text{for each } x \in A.$$

For finite strings $\boldsymbol{a}, \boldsymbol{b} \in A$, we define $\boldsymbol{a} \leqslant_d \boldsymbol{b}$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same order relation, i.e. $\forall i, j\, [a_i < a_j \Leftrightarrow b_i < b_j]$ and $\forall i, j\, [d(a_i, a_j) \leqslant d(b_i, b_j)$ and $d(\mathbf{0}, a_i) \leqslant d(\mathbf{0}, b_i)$ and $d(a_i, \mathbf{1}) \leqslant d(b_i, \mathbf{1})]$.

Note that if d is a recursive function, then $\leqslant_d$ is a recursive relation.

The crucial feature for applying the theorem is:

Contraction Lemma. *In any linear ordering, if $\boldsymbol{b}$ satisfies a universal formula $\phi(\boldsymbol{x})$, and $\boldsymbol{a} \leqslant_d \boldsymbol{b}$, then $\boldsymbol{a}$ also satisfies ϕ.*

Conversely, if $\boldsymbol{a} \not\leqslant_d \boldsymbol{b}$, then there is a universal formula ϕ such that $\boldsymbol{b}$ satisfies ϕ but $\boldsymbol{a}$ does not.

Proof. We prove the contrapositive. Suppose we have $\boldsymbol{a} \leqslant_d \boldsymbol{b}$ and existential ϕ such that $\phi[\boldsymbol{a}]$. Now $\phi(\boldsymbol{x})$ is logically equivalent to a formula $\exists \boldsymbol{y}\, \psi(\boldsymbol{x}, \boldsymbol{y})$ when ψ is quantifier-free. Since $\phi[\boldsymbol{a}]$, there are $\boldsymbol{c}$ such that $\psi[\boldsymbol{a}, \boldsymbol{c}]$. Because $\boldsymbol{a} \leqslant_d \boldsymbol{b}$, there are $\boldsymbol{c}'$ such that $\boldsymbol{a}, \boldsymbol{c} \leqslant_d \boldsymbol{b}, \boldsymbol{c}'$. Since $\boldsymbol{b}, \boldsymbol{c}'$ are in the same order relationship as $\boldsymbol{a}, \boldsymbol{c}$, they satisfy the same quantifier-free formulae. In particular we must have $\psi[\boldsymbol{b}, \boldsymbol{c}']$. So $\phi[\boldsymbol{b}]$.

For the converse, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are not in the same order relation, then there is a quantifier-free formula satisfied by $\boldsymbol{b}$ but not $\boldsymbol{a}$.

Otherwise, we must have $d(a_i, a_j) > d(b_i, b_j) = n$ for some i and j, with say $b_i < b_j$. Then $\boldsymbol{b}$ satisfies the formula

$$\forall y_1 \cdots y_n \left[\bigwedge_{i \leqslant k \leqslant n} (x_i < y_k < x_j) \rightarrow \bigvee_{\substack{k \neq l \\ 1 \leqslant k, l \leqslant n}} (y_k = y_l) \right],$$

but $\boldsymbol{a}$ does not.

Consider the following condition:

Condition (0). For every finite list of parameters $\boldsymbol{c} \in A$, there is $l \in L$ such that for every finite list $\boldsymbol{m} \in A$, there are $l' \notin L$ and $\boldsymbol{m}'$ with $\boldsymbol{c}, l', \boldsymbol{m} \leqslant_d \boldsymbol{c}, l, \boldsymbol{m}$.

If condition (0) holds, then L cannot be formally Σ_2^0. For if it is with parameters $\boldsymbol{c}$, let l be the element associated with $\boldsymbol{c}$ by condition (0).

Now L is equivalent to a disjunction of $\exists \bigwedge\!\!\!\bigwedge \forall$ formulae, one of which, ϕ say, must be satisfied by l. But, by the Contraction Lemma, we see that ϕ is also satisfied by $l' \notin L$, which is impossible.

Proposition 2. *In a recursive linear order, L is formally Σ_2^0 iff there is a finite list of parameters $\mathbf{0} = c_0 < c_1 < \cdots < c_n < c_{n+1} = \mathbf{1}$ such that each interval (c_i, c_{i+1}) is of the type $k \cdot \eta$ (for some $k_i \in \mathbb{N}$) or contains no element of L.*

Proof. If $A = k \cdot \eta$, then

$$x \in L \Leftrightarrow \exists y_1 \cdots y_k \, \forall z \left[x = y_1 < y_2 < \cdots < y_k \,\&\, \bigwedge_i \neg(y_i < z < y_{i+1}) \right].$$

So if every interval (c_i, c_{i+1}) is of this type or contains no left limit point, then L is formally Σ_2^0.

If not, then for any $\boldsymbol{c}$, there is an interval $I = (c_i, c_{i+1})$ containing a left limit point, l, but not of type $k \cdot \eta$. We can assume that $d(c_i, l) = d(c_{i+1}, l) = \infty$ for if not, consider a new sequence $\boldsymbol{c}$ including the finitely many points between l and c_i or c_{i+1}.

Now either (a) I contains an infinite block, (i.e. of type ω, ω^* or $\omega^* + \omega$),

or (b) I contains blocks of arbitrarily large finite size,

or (c) I contains infinitely many blocks of maximal length (k say), and a block of length $<k$.

We want to show that condition (0) holds.

Cases (a) & (b). Take l to be any limit point infinitely distant from c_i and c_{i+1}. The procedure for finding l' and $\boldsymbol{m}'$ is as follows: Let $\boldsymbol{m}_1$ consist of those elements of $\boldsymbol{m}$ which are finitely distant from c_i or c_{i+1}. So $\boldsymbol{m} = \boldsymbol{m}_1, \boldsymbol{m}_2$, say. Let n be the number of elements in $\boldsymbol{m}_2$.

Find a segment of length $n+2$, which does not include any of the $\boldsymbol{m}_1$, and let $l', \boldsymbol{m}_2^*$ be the last $n+1$ elements of this segment and such that they correspond in order to $l, \boldsymbol{m}_2$. Then clearly

$$c_i, c_{i+1}, l', \boldsymbol{m}_1, \boldsymbol{m}_2^* \leqslant_d c_i, c_{i+1}, l, \boldsymbol{m}_1, \boldsymbol{m}_2.$$

Case (c). Note that every element, except possibly finitely many, which we may treat as if they were in $\boldsymbol{c}$, is infinitely distant from c_i and c_{i+1}. Let l be the left-most element of some block of length $<k$.

Given $\boldsymbol{m}$, proceed as follows: Let n be the number of elements in $\boldsymbol{m}$. Find $n+1$

blocks of length k. Then let $l', \boldsymbol{m}'$ be elements of those blocks so that they correspond in order to l, $\boldsymbol{m}$, and l' is not a left limit point. This is possible because l is in a block of length $<k$. Furthermore, since k is the maximum block size, this can be done so that

$$c_i, c_{i+1}, l', \boldsymbol{m}' \leqslant_d c_i, c_{i+1}, l, \boldsymbol{m}. \qquad \square$$

Now a linear order contains no left limit points iff each element has an immediate predecessor or is the leftmost element. This is the same as saying that every block (except possibly the leftmost) is of type ω or $\omega^* + \omega$. The leftmost block, if it exists, could also be finite or of type ω.

So a countable linear order contains no left limit points iff it is γ, $k + \gamma$ or $\omega + \gamma$, where γ is a linear order obtained by taking an arbitrary countable linear order and replacing each point by a block of type either ω^* or $\omega^* + \omega$. So we may say:

L is formally Σ^0_2 iff $\mathfrak{A}$ can be subdivided at points into finitely many segments of type $k \cdot \eta$, γ or $\omega + \gamma$. [$k + \gamma$ is treated by including the k part amongst the subdividing points.]

By applying the above argument 'from right to left' we see that R is formally Σ^0_2 iff $\mathfrak{A}$ can be subdivided at points into finitely many segments of types $k \cdot \eta$, γ^* or $\gamma^* + \omega^*$.

This argument also applies for T, as $t' \notin L \Rightarrow t' \notin T$. T is formally Σ^0_2 iff $\mathfrak{A}$ can be subdivided at points into finitely many segments, each of which contains no two-sided limit points or is of type η.

In order to apply Theorem 1, we need some extra decidability. We will assume that we are given a recursive linear order with a recursive distance function (defined on $(A \cup \{\mathbf{0}, \mathbf{1}\})^2$, for which L (or R or T) is recursive.

Then the existential diagram of $\mathfrak{A}$ is recursive, as $\exists x\, (a < x < b) \Leftrightarrow d(a, b) \geqslant 1$, and the Contraction Lemma shows that $\boldsymbol{a} \leqslant_1 \boldsymbol{b} \Leftrightarrow \boldsymbol{b} \leqslant_d \boldsymbol{a}$, so this relation is recursive. So conditions (A) and (B) are satisfied. Condition (C) is implied by condition (0) provided that the l can be found effectively, so we assume this as well. This can easily be arranged in the case of 'natural' linear orders like these examples:

Examples.

Ordering	*Intrinsically* Σ^0_2	*Not intrinsically* Σ^0_2
(i) Constructive well ordering $\geqslant \omega^2$	R, T	L
(ii) $\omega + \eta$, $\eta + \omega^*$		L, R, T
(iii) $\eta + k \cdot \eta$ $(k \neq 1)$		L, R, T
(iv) $k_1 \cdot \eta + k_2 \cdot \eta$ $(1 \neq k_1 \neq k_2 \neq 1)$	T	L, R
(v) $\eta + \omega$, $\omega^* + \eta$	L, R, T	
(vi) $\eta + 1 + 2 \cdot \eta$	L, R, T	

Ordering	Intrinsically Σ_2^0	Not intrinsically Σ_2^0
(vii) $(k+\eta)\cdot\alpha$ ($k \geqslant 2$, $\omega \leqslant \alpha$ a constructive ordinal		L, R, T
(viii) The order obtained by replacing each point, p, on the rational line by a block of size $\lvert a \rvert$, where $p = a/b$ reduced to lowest terms.		L, R, T

Proof. In (i) and (iv) T is intrinsically Σ_2^0 because it is empty. The same applies to R in (i).

For (v), let a be the first element of the ω part of $\eta+\omega$. Then $x \in L$ iff $x \leqslant a$, and $x \in R = T$ iff $x < a$.

$\omega^* + \eta$ is similar.

For (vi), let a be the point between the η and $2\cdot\eta$ parts. Then $x \in T$ iff $x \leqslant a$. And $x \in L$ iff $\exists y\ \forall z\ [(x \leqslant a)$ or $(x < y)\ \&\ \neg(x < z < y)]$. And $x \in R$ iff $\exists y\ \forall z\ [(x \leqslant a)$ or $(y < x)\ \&\ \neg(y < z < x)]$.

In the other cases we wish to apply Theorem 1. In each case it is clear that we can arrange for the necessary decidability in our numbering of the ordering. So we want to show that condition (0) holds.

We suppose that all lists of parameters $\boldsymbol{c}$ include the points $\mathbf{0}$ and $\mathbf{1}$.

(i) Given $\boldsymbol{c}$, let c_a be the last parameter in the initial ω^2 segment, and c_b the first parameter after the initial ω^2 segment. Then (c_a, c_b) is a constructive ordinal of type $\geqslant \omega^2$, and so certainly contains a left limit point infinitely distant from c_a and c_b, and a block of type ω.

(ii) For $\omega+\eta$, given $\boldsymbol{c}$ let c_a be the last parameter in the ω part, and c_b be the first parameter in the η part. Then (c_a, c_b) contains a limit point (in the η part) and a block of type ω.

(iii) Given $\boldsymbol{c}$, let c_a be the last parameter in the η part, and c_b be the first parameter in the $k\cdot\eta$ part. Then (c_a, c_b) satisfies case (c) in the proof of the proposition.

(iv) is similar to (iii).

(vii) Given $\boldsymbol{c}$, let c_a be the last parameter in the initial $(k+\eta)\cdot\omega$ segment, and c_b be the first parameter after that segment. Then (c_a, c_b) contains a segment of type $(k+\eta)\cdot\omega$. This satisfies part (c) above, as there are infinitely many blocks of size k here.

(viii) Given $\boldsymbol{c}$, let c_a be the last parameter associated with a negative number and c_b be the first parameter associated with a positive number. Then (c_a, c_b) contains a two-sided limit (associated with $1/n$ for sufficiently large n), and arbitrarily long finite blocks. $\square$

5. Examples: boolean algebras

Countable boolean algebras may be thought of as being generated by a chain in the partial order of the algebra. That is, associated with each countable linear order with initial point, α, there is a boolean algebra, $B(\alpha)$. The elements of $B(\alpha)$ may be identified with finite unions of semi-open intervals (of the form $[a, b)$, where b may be ∞) from α. The operations are then normal set union, intersection and complementation. [In fact, all countable boolean algebras may be represented this way.]

So we will be interested in recursive boolean algebras that may be presented as $B(\alpha)$ for some recursive linear order, α, with recursive distance function.

For example, $B(\omega)$ is the boolean algebra consisting entirely of finite joins of its countably many atoms, and their complements.

The Frechet ideal, F, of a boolean algebra is the set of finite joins of atoms. It may be represented

$$x \in F \Leftrightarrow \bigvee\!\!\!\bigvee_{n\in\omega} \exists y_1 \cdots y_n\, \forall z \left[x = y_1 \vee \cdots \vee y_n \;\&\; \bigwedge_{1\leqslant i\leqslant n} \neg(0 < z < y_i)\right]$$

So F is formally Σ^0_2. We will see that it is not necessarily intrinsically Π^0_2.

The atomless ideal, A, of a boolean algebra is defined by

$$x \in A \Leftrightarrow \forall y\, \exists z\, [0 < y \leqslant x \rightarrow 0 < z < y].$$

That is, A is the set of all those elements with no atoms below them. We see that A is formally Π^0_2. It will turn out that it is not necessarily intrinsically Σ^0_2.

Before we can apply the theorem, we need to know when $\boldsymbol{a} \leqslant_1 \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in B(\alpha)$.

Given $\boldsymbol{a} = a_1, \ldots, a_n \in B(\alpha)$, the atoms of the subalgebra generated by $\boldsymbol{a}$ are of the form $a_1^{(\prime)} \wedge a_2^{(\prime)} \wedge \cdots \wedge a_n^{(\prime)}$, where the complements may or may not be taken individually. Of course, some of these expressions could be equal to 0. Let $a_1^*, a_2^*, \ldots, a_{2^n}^*$ be these expressions taken in some pre-determined order. We have the following:

Lemma 4. $\boldsymbol{a} \leqslant_1 \boldsymbol{b}$ *iff for each* i

(i) a_i^*, b_i^* *are both* 0, *or*

(ii) a_i^*, b_i^* *are both non-zero, and* $\#(a_i^*) \geqslant \#(b_i^*)$, *where* $\#(a)$ *is the number of points in the intervals comprising* a *in* α.

Proof. Suppose there is some i for which neither (i) nor (ii) holds.

If $a_i^* = 0$ and $b_i^* \neq 0$, then the $\boldsymbol{b}$ do not satisfy some formula of the form

$$x_1^{(\prime)} \wedge \cdots \wedge x_n^{(\prime)} = 0$$

which is satisfied by the $\boldsymbol{a}$.

If $a_i^* \neq 0$ and $b_i^* = 0$, there is such a formula of the form

$$x_1^{(\prime)} \wedge \cdots \wedge x_n^{(\prime)} \neq 0.$$

If $k-1=\#(a_i^*)<\#(b_i^*)$, then some formula of the form

$$\forall y_1\cdots y_k\left[\bigwedge_{1\leqslant j\leqslant k}(0<y_i\leqslant x_1^{(\prime)}\wedge\cdots\wedge x_n^{(\prime)})\rightarrow\bigvee_{\substack{1\leqslant j,m<k\\ j\neq m}}(y_j\vee y_m\neq 0)\right]$$

is satisfied by $\boldsymbol{a}$, but not by $\boldsymbol{b}$.

Conversely, if (i) or (ii) holds for each i, then given $\boldsymbol{d}$, partition the $\boldsymbol{b}^*$ into intervals, various unions of which form the atoms of the subalgebra generated by $\boldsymbol{b}, \boldsymbol{d}$. But then we can partition the $\boldsymbol{a}^*$ in a corresponding way (because there are enough points below them). That is, there exist $\boldsymbol{c}$ so that the subalgebras generated by $\boldsymbol{a}, \boldsymbol{c}$ and $\boldsymbol{b}, \boldsymbol{d}$ are isomorphic. So $\boldsymbol{a}\leqslant_1\boldsymbol{b}$. $\square$

If α is a recursive linear order with recursive distance function, then this lemma implies that condition (B) of Theorem 1 is satisfied for $B(\alpha)$. Also $B(\alpha)$ will have a recursive existential diagram, and F will be recursive. [$a\in F\Leftrightarrow\#(a)$ is finite.]

In fact, we wish to apply the theorem to the complement, F', of F. We will do this for $B(\alpha)$ in the cases where α is an ordinal $\geqslant\omega^2$, where $\alpha=\omega+\eta$ and where $\alpha=2\cdot\eta$.

It remains to show that condition (C) is satisfied in each case, which will also imply that F' is not formally Σ_2^0.

Any list of parameters, $\boldsymbol{p}$, partitions α into finitely many disjoint intervals.

In the case where α is an ordinal $\geqslant\omega^2$, one of these intervals, I say, will be of order $\geqslant\omega^2$. Take a to be the first sub-interval of type ω of I. Then any $\boldsymbol{b}$, together with a, gives a finite partition of I. Let $a', \boldsymbol{b}'$ give a corresponding partition of I in which all the intervals except the last are shrunk to singletons. Then $a, \boldsymbol{b}\leqslant_1 a', \boldsymbol{b}'$, and $a'\in F$.

In the case $\alpha=\omega+\eta$, we do the same thing, except that we take I to be the interval of type $\omega+\eta$, and a to be an interval inside the η part of I which does not have the same end-point as I.

In the case $\alpha=2\cdot\eta$, we take I to be any infinite interval in the partition given by $\boldsymbol{p}$, and a to be any infinite interval inside I. Then any $\boldsymbol{b}$, together with a, gives a finite partition of I. If in this partition a is divided into k intervals, then take a' to be the union of k atoms within I, and $\boldsymbol{b}'$ to make these atoms correspond to the intervals that a is divided into by $\boldsymbol{b}$, and also to ensure that atoms in the $\boldsymbol{b}$ partition of I correspond to atoms in the $\boldsymbol{b}'$ partition. Then we have
$a, \boldsymbol{b}\leqslant_1 a', \boldsymbol{b}$, and $a'\in F$, as required.

So in these three cases we can apply Theorem 1 to obtain:

Example. There are recursive boolean algebras isomorphic to $B(\alpha)$ for α an ordinal $\geqslant\omega^2$, $B(\omega+\eta)$ and $B(2\cdot\eta)$ in which the Frechet ideal, F, is not Π_2^0.

In fact, every super-atomic countable boolean algebra (i.e. one with no atomless subalgebra) is of the form $B(\alpha)$ for some countable ordinal α. So we will have dealt with all super-atomic recursive boolean algebras after considering $B(\alpha)$ for α an ordinal $<\omega^2$. Suppose $\alpha = \omega \cdot k + n$. Let $\boldsymbol{p}$ consist of the k ω-intervals and the final interval. Then

$$x \in F \Leftrightarrow \bigwedge_{1 \leqslant i \leqslant k+1} (x \wedge p_i \in F) \Leftrightarrow \bigwedge_{1 \leqslant i \leqslant k+1} (x' \wedge p_i \in F').$$

So in this case, since F' is formally Π^0_2, F is formally Π^0_2.

In $B(\omega + \eta)$, the atomless ideal, A, consists of unions of intervals entirely within the η part. In this case we can arrange that A is recursive. Condition (C) is satisfied in exactly the same way as above ($a \in A$, and $a' \notin A$), so we have:

Example. There is a recursive boolean algebra isomorphic to $B(\omega + \eta)$ in which the atomless ideal, A, is not Σ^0_2.

6. Examples: reduced abelian p-groups

A reduced abelian p-group (RAP) is an abelian group in which the order of each element is a power of p (a prime) and which contains no divisible sub-group. That is, there is no sequence of elements $y_1, y_2, \ldots$ such that $py_{i+1} = y_i$. For a recursive RAP, we merely require that the group multiplication be recursive (on a recursive set of elements). Note that as the order of each element is finite, we can find inverses recursively by successively calculating x, px, $p^2x, \ldots, p^n$ until $p^n x = 0$. Then $(-x) = (p^n - 1)x$.

For a RAP G, define G_α as follows: $G_0 = G$, $G_{\alpha+1} = pG_\alpha$, and if α is a limit ordinal, $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$.

Define $P = \{x \in G \mid px = 0\}$ and $P_\alpha = P \cap G_\alpha$. The height of an element $x \neq 0$, $h(x)$, is defined to be the unique α where $x \in G$, $x \notin G_{\alpha+1}$. Conventionally $h(0) = \infty > h(x)\ \forall x \in G$, $x \neq 0$.

For example consider the RAP given by the generators $x, y_1, y_2, y_3, \ldots$ and the relations $px = 0$, $x = py_1 = p^2y_2 = \cdots = p^n y_n = \cdots$. Here $h(x) = \omega$.

In this example there are only p elements of height $\geqslant \omega$ (the multiples of x), but we will consider the direct sum of countably many copies of this group:

Let the RAP E be given by the generators

$$x_i,\ y_{ij} \qquad (1 \leqslant i, j < \omega)$$

and the relations

$$px_i = 0, \quad p^j y_{ij} = x_i \qquad (1 \leqslant i, j < \omega).$$

We will consider

$$I = \{\text{elements of } G \text{ of infinite height}\} = \left\{x \;\middle|\; \bigwedge\!\!\!\bigwedge_{n<\omega} \exists y\, [p^n y = x]\right\}.$$

In E, I is exactly the set of all linear combinations of the x_i.

We see from the definition that I is formally Π_2^0. We will show that in this case I is not intrinsically Σ_2^0.

First we need to prove a general lemma.

Consider a RAP G with infinitely many elements of infinite height. (So P_ω is infinite.) Let $\boldsymbol{a}$, $\boldsymbol{b}$ be finite strings of elements of G. Let $\langle \boldsymbol{a} \rangle$ denote the (finite) subgroup of G generated by $\boldsymbol{a}$. Write $\boldsymbol{a} \leqslant_h \boldsymbol{b}$ if there is an isomorphism $f : \langle \boldsymbol{a} \rangle \cong \langle \boldsymbol{b} \rangle$ such that $f(a_i) = b_i$ and $h(f(a)) \geqslant \min\{\omega, h(a)\}$ for every $a \in \langle \boldsymbol{a} \rangle$. (There is at most one isomorphism, as $f(a_i) = b_i$ for each i.)

Lemma 5. *If $\boldsymbol{a} \leqslant_h \boldsymbol{b}$ and $\boldsymbol{b}$ satisfies a universal formula ϕ, then $\boldsymbol{a}$ also satisfies ϕ.*

Proof. This is equivalent to saying that for every finite extension of $\langle \boldsymbol{a} \rangle$ (that is, a finite group containing $\langle a \rangle$), there is an isomorphic extension of $\langle \boldsymbol{b} \rangle$.

So suppose we are given finite subgroups $A = \langle \boldsymbol{a} \rangle$, B of G, a height increasing isomorphism $f : A \cong B$, and a finite extension A' of A. We want to extend f to an isomorphism on A'.

We will make this extension in finitely many stages. At each stage we will take $x \in A' - A$ with $px \in A$ and $h(x) \geqslant h(x + a)$ for each $a \in A$, and extend f to $A_1 = \langle x, \boldsymbol{a} \rangle$.

Since f is height increasing, we have

$$(*) \quad h(f(a)) \geqslant \min\{N, h(a)\} \quad \text{for each } a \in A \text{ where } N = \operatorname{card}(A' - A) < \omega.$$

We will ensure that when we extend f it still satisfies $(*)$ with $N = \operatorname{card}(A' - A_1)$. This assures us that we will be able to carry out all the finitely many stages necessary to extend the isomorphism to A', as $\operatorname{card}(A' - A) \geqslant$ height relative to A' of every element of $A' - A$.

So, given $x \in A' - A$ with $px \in A$ and $h(x)$ maximal in $A' - A$, let $f(px) = y$. Then $h(y) \geqslant \min\{N, h(px)\} \geqslant \min\{N, h(x) + 1\}$.

So there is w with $pw = y$ such that $h(w) \geqslant \min\{N - 1, h(x)\}$.

If $w \notin B$, take $z = 0$. Otherwise, let $z \in P_\omega - B$. Such a z exists, since P_ω is infinite.

Then, in both cases, $w + z \notin B$ and $p(w + z) = pw + pz = y + 0 = y$. Also

$$\begin{aligned} h(w + z) &\geqslant \min\{h(w), h(z)\} \\ &\geqslant \min\{N - 1, h(x), \omega\} = \min\{N - 1, h(x)\}. \end{aligned}$$

So extend f to $A_1 = \langle x, \boldsymbol{a} \rangle$ as follows:

$$f(rx + a) = r(w + z) + f(a) \quad \text{for } 0 \leqslant r < p \text{ and } a \in A.$$

It remains to check that f still satisfies $(*)$. Note that $N - 1 \geqslant \operatorname{card}(A' - A_1)$. Now, if $0 < r < p$,

$$\begin{aligned} h(r(w + z) + f(a)) &\geqslant \min\{h(w + z), h(f(a))\} \\ &\geqslant \min\{N - 1, h(x), h(a)\}, \end{aligned}$$

and $h(rx+a)=\min\{h(x), h(a)\}$, since $h(x)$ was maximal in $A'-A$. Hence

$$h(r(w+z)+f(a)) \geqslant \min\{N-1, h(rx+a)\},$$

and so f still satisfies $(*)$. $\square$

Lemma 6. *If $\boldsymbol{a} \not\leqslant_h \boldsymbol{b}$, there is some universal formula ϕ such that $\boldsymbol{b}$ satisfy ϕ but $\boldsymbol{a}$ do not.*

Proof. If $\langle \boldsymbol{a}\rangle \not\equiv \langle \boldsymbol{b}\rangle$, then there is a quantifier free formula satisfied by $\boldsymbol{b}$ but not by $\boldsymbol{a}$. So suppose we have $f:\langle \boldsymbol{a}\rangle \cong \langle \boldsymbol{b}\rangle$, but f is not height increasing. So there is $a \in \langle \boldsymbol{a}\rangle$ such that $m=h(f(a))<\min\{\omega, h(a)\}$. Clearly, $\langle \boldsymbol{a}\rangle$ can be extended by adding a (p^{m+1})-th root of a, but we cannot add a corresponding (p^{m+1})-th root of $f(a)$ to $\langle \boldsymbol{b}\rangle$. $\square$

The above lemmas give us:

$$\boldsymbol{a} \leqslant_1 \boldsymbol{b} \Leftrightarrow \boldsymbol{b} \leqslant_h \boldsymbol{a}.$$

If we assume that the function $h': G \to \mathbb{N} \cup \{\infty\}$ defined by

$$h'(x)=\begin{cases} h(x) & \text{if } h(x)<\omega, \\ \infty & \text{if } h(x)\geqslant\omega \end{cases}$$

is recursive, then we see that $\leqslant_h$ is a recursive relation, and the existential diagram of G is recursive.

Thus, given a recursive RAP G for which the function h' is recursive and the set I is infinite and recursive, it remains only to check condition (C) before we can apply the theorem to I. We will do this for the group E given above.

Given $\boldsymbol{c}$, take $x=x_{i_0} \notin \langle \boldsymbol{c}\rangle$.

Now given $\boldsymbol{a}$, let $n=\operatorname{card}(\langle \boldsymbol{c}, x, \boldsymbol{a}\rangle)$. Let j_0 be such that $(p^0 y_{j_0(n+1)} - p^{n+1} y_{j_0(n+2)}) \notin \langle \boldsymbol{c}, x, \boldsymbol{a}\rangle$, and put $z=y_{j_0(n+1)}-p y_{j_0(n+2)}$. Then define a map f on $\langle \boldsymbol{c}, x, \boldsymbol{a}\rangle$ as follows: For $i \neq i_0$,

$$\begin{cases} x_i \mapsto x_i, \\ y_{ik} \mapsto y_{ik} & \text{for each } k \end{cases}$$

and

$$\begin{cases} x_{i_0} \mapsto x_{i_0}+p^n z, \\ y_{i_0k} \mapsto y_{i_0k}+p^{n-k} z & \text{for each } k. \end{cases}$$

This is well defined because

$$\begin{aligned} p(x_{i_0}+p^n z) &= p x_{i_0} + p^{n+1} y_{j_0(n+1)} - p^{n+2} y_{j_0(n+2)} \\ &= 0 + x_{j_0} - x_{j_0} = 0 \end{aligned}$$

and

$$p^k(y_{i_0k}+p^{n-k}z) = p^k y_{i_0k} + p^n z = x_{i_0} + p^n z,$$

and also because of the choice of n. That is, y_{i_0k} with $k > n$ cannot be in $\langle \boldsymbol{c}, x, \boldsymbol{a} \rangle$.

Clearly f is a height-decreasing isomorphism, and $h(f(x_{i_0})) = n < \omega$.

Thus, given $\boldsymbol{c}$ we have found $x \in I$ such that given $\boldsymbol{a}$, we can find $x' = f(x)$ and $\boldsymbol{a}' = f(\boldsymbol{a})$ such that $\boldsymbol{c}, x', \boldsymbol{a}' \leqslant_h \boldsymbol{c}, x, \boldsymbol{a}$.

The existence of such an $x \in I$ for every choice of parameters $\boldsymbol{c}$ also shows that I is not formally Σ_2^0.

Clearly we can number E so that in the course of the construction we can make a list of the i for which we have used some x_i or y_{ij} by stage s. Then we can choose x effectively by choosing it to be x_{i_0} for some i_0 not on the list at that stage.

So we have the following:

Example. There is a recursive RAP isomorphic to E in which the set of elements of infinite height is not Σ_2^0. [E is given by the generators x_i, y_{ij} ($1 \leqslant i, j < \omega$) and the relations $px_i = 0$, $p^j y_{ij} = x_i$ ($1 \leqslant i, j < \omega$).]

7. Conditions for intrinsic relations

For the general case, we need to generalize the relation $\leqslant_1$.

Definition. $\boldsymbol{a} \leqslant_1 \boldsymbol{b}$ if every finitary universal formula true for $\boldsymbol{a}$ in $\mathfrak{A}$ is true for $\boldsymbol{b}$.

$\boldsymbol{a} \leqslant_{\beta+1} \boldsymbol{b}$ if for each sequence $\boldsymbol{d}$ there exists a sequence $\boldsymbol{c}$ such that $\boldsymbol{a}, \boldsymbol{c} \geqslant_\beta \boldsymbol{b}, \boldsymbol{d}$.

If α is a limit ordinal, $\boldsymbol{a} \leqslant_\alpha \boldsymbol{b}$ if $\boldsymbol{a} \leqslant_\beta \boldsymbol{b}$ for all $\beta < \alpha$.

Definition. Given a recursive structure $\mathfrak{A}$ and a new relation R on $\mathfrak{A}$, we define for each finite sequence $\boldsymbol{p} \in |\mathfrak{A}|$ and each ordinal $\alpha \geqslant 2$ the subset $\mathrm{Rcl}_\alpha(\boldsymbol{p})$ of R.

If $\alpha = \beta + 1$, then $x \in \mathrm{Rcl}_\alpha(\boldsymbol{p})$ if for some $\boldsymbol{a}$, whenever $\boldsymbol{p}, x, \boldsymbol{a} \leqslant_\beta \boldsymbol{p}, x', \boldsymbol{a}'$ then $x' \in R$.

If α is a limit ordinal, then $\mathrm{Rcl}_\alpha(\boldsymbol{p}) = \bigcup_{\beta<\alpha} \mathrm{Rcl}_\beta(\boldsymbol{p})$.

The following lemmas establish the connection between these relations and infinitary formulae.

Lemma 7. *Suppose the relations $\leqslant_\beta$ (for $\beta \leqslant \alpha$) for $\mathfrak{A}$ are uniformly recursive, and the existential diagram of $\mathfrak{A}$ is recursive. Then we can effectively find from $\boldsymbol{a} \in |\mathfrak{A}|$ (a Gödel number for) a recursive Π_α formula $\varphi_\alpha^{\boldsymbol{a}}$ such that for all $\boldsymbol{b} \in |\mathfrak{A}|$, $\boldsymbol{a} \leqslant_\alpha \boldsymbol{b}$ iff $\mathfrak{A} \models \varphi_\alpha^{\boldsymbol{a}}[\boldsymbol{b}]$.*

The proof is a simple extension of that given in [1]. □

Lemma 8. *Given $\alpha \geqslant 2$, suppose the relations $\leqslant_\beta$ (for $\beta < \alpha$) for $\mathfrak{A}$ are uniformly recursive, and the existential diagram of $\mathfrak{A}$ and R are recursive. If, for some $\boldsymbol{p}$, $\mathrm{Rcl}_\alpha(\boldsymbol{p}) = R$, then R is formally Σ_α^0.*

Proof. We first deal with the successor ordinal case $\alpha = \beta + 1$. We have $R = \mathrm{Rcl}_\alpha(\boldsymbol{p})$, so for each $r \in R$ there is $\boldsymbol{a}$ such that whenever $\boldsymbol{p}, r, \boldsymbol{a} \leqslant_\beta \boldsymbol{p}, r', \boldsymbol{a}'$ then $r' \in R$. This relation between $\boldsymbol{p}, r, \boldsymbol{a}$ is Π^0_1, so some such $\boldsymbol{a}$, depending on r, can be found by a Δ^0_2 process.

Let $\varphi_\beta^{\boldsymbol{p}, r, \boldsymbol{a}_r}(\boldsymbol{x}, y, \boldsymbol{z}_r)$ be the Π_β formula guaranteed by Lemma ~~2~~ 7. Then

$$\mathfrak{A} \models y \in R \leftrightarrow \bigvee\!\!\!\bigvee_{r \in R} \exists \boldsymbol{z}_r\, \varphi_\beta^{\boldsymbol{p}, r, \boldsymbol{a}_r}(\boldsymbol{p}, y, \boldsymbol{z}_r).$$

This disjunction in this formula is ~~recursive~~ Σ^0_2, so R has a formally $\Sigma^0_{\cancel{2}\alpha}$ definition.

Now suppose α is a limit ordinal. Then for each $r \in R$, there is $\beta < \alpha$ and $\boldsymbol{a}$ such that whenever $\boldsymbol{p}, r, \boldsymbol{a} \leqslant_\beta \boldsymbol{p}, r', \boldsymbol{a}'$ then $r' \in R$. This relation between $\boldsymbol{p}, r, \boldsymbol{a}$ and (a notation for) β is Π^0_1, so some such β and $\boldsymbol{a}$, depending on r, can be found by a Δ^0_2 process. Then

$$\mathfrak{A} \models y \in R \leftrightarrow \bigvee\!\!\!\bigvee_{r \in R} \exists \boldsymbol{z}_r\, \varphi_{\beta_r}^{\boldsymbol{p}, r, \boldsymbol{a}_r}(\boldsymbol{p}, y, \boldsymbol{z}_r)$$

and this disjunction is clearly a Σ_α formula. $\square$

8. Recursive labelling systems

We now give the underlying definitions we need to apply the general theorem of C. J. Ash. We are following [2] closely, and the reader is referred there for motivation and explanation.

A *recursive labelling system* (T, L, S, N, F) on a complete recursive metric space X with basic open sets $B(X)$, consists of a recursive tree T; an r.e. set of labels L for the nodes of the tree; an r.e. relation $S \subset T \times L$ with $(u, l) \in S$ intended to mean that l is a suitable label for the node u; an r.e. relation $N \subset T \times L \times T \times L$ with $(u, l, u', l') \in N$ intended to mean that l' is a correct next label for the successor node u' of u if u is labelled by l; and a function $F: L \to \mathscr{P}(X)$ for which the relation $F(l) \cap \sigma \neq \emptyset$ on $L \times B(X)$ is r.e.

A *correct labelling* of the path $u_0, u_1, \ldots$ in T is a sequence $l_0, l_1, \ldots$ from L such that for each n, $S(u_n, l_n)$ and $N(u_n, l_n, u_{n+1}, l_{n+1})$.

An *adherent point* of a correct labelling $\{l_n\}$ of an infinite path $\{u_n\}$ is a point $x \in X$ such that for every open set U containing x, $F(l_n) \cap U \neq \emptyset$ for sufficiently large n.

An *α-system* (for α a successor ordinal) is a recursive labelling system together with a family $\{\lhd_\beta \mid 1 \leqslant \beta < \alpha\}$ of uniformly r.e. binary relations on L satisfying the following conditions:

(1) If u_0 is the root node of T, then for any $\sigma \in B(X)$ there exists $l \in L$ such that $S(u_0, l)$ and $F(l) \cap \sigma \neq \emptyset$.

(2) If $S(u, l)$ and $N(u, l, v, m)$, then $S(v, m)$.

(3) Each $\lhd_\beta$ is reflexive and transitive.

(4) If $N(u, l, v, m)$, then $l \lhd_{\alpha-1} m$.

(5) If $1 \leqslant \gamma_1 < \gamma_2 < \alpha$ and $l \lhd_{\gamma_2} m$, then $l \lhd_{\gamma_1} m$.

(6) If $l \lhd_1 m$, then $F(l) \supseteq F(m)$.

(7) Suppose that $S(u, l)$ and v is a successor node of u. Suppose also that $\alpha > \alpha_k > \cdots > \alpha_1 > \alpha_0 \geqslant 1$ and that $l = l_k \lhd_{\alpha_k} \cdots \lhd_{\alpha_2} l_1 \lhd_{\alpha_1} l_0$ where $F(l_0) \cap \sigma \neq \emptyset$. Then there exists m for which $N(u, l, v, m)$, $F(m) \cap \sigma \neq \emptyset$, and $l_i \lhd_{\alpha_i} m$ (for $i = 0, 1, \ldots, k-1$).

If α is a limit ordinal, then there is a recursive sequence $\langle \gamma_n \rangle$ of successor ordinals with limit α.

A $\langle \gamma_n \rangle$*-path* in a recursive tree is an infinite path $u_0, u_1, \ldots$ such that for some recursive sequence $e_0, e_1, \ldots$, each e_n is a $\Delta^0_{\alpha_n}$ index for u_{n+1}.

For α a limit ordinal, an α*-system* has the same definition as above, except that (4) and (7) are replaced by:

(4′) If $N(u, l, v, m)$ and u is a node of level n, then $l \lhd_{\gamma_n - 1} m$.

(7′) Suppose that $S(u, l)$, u is a node of level n and v is a successor of u. Suppose also that $\gamma_n > \alpha_k > \cdots > \alpha_1 > \alpha_0 \geqslant 1$ and that $l = l_k \lhd_{\alpha_k} \cdots \lhd_{\alpha_2} l_1 \lhd_{\alpha_1} l_0$ where $F(l_0) \cap \sigma \neq \emptyset$. Then there exists m for which $N(u, l, v, m)$, $F(m) \cap \sigma \neq \emptyset$ and $l_i \lhd_{\alpha_i} m$ (for $i = 0, 1, \ldots, k-1$).

A *path-generating instruction* in a recursive labelling system is a partial function $p: T \times L \to T$ mapping each (u, l) for which $S(u, l)$ to some successor of u.

A *labelling* of an instruction p is an infinite path $u_0, u_1, \ldots$ together with a correct labelling $l_0, l_1, \ldots$ of this path such that $p(u_n, l_n) = u_{n+1}$ for each n.

If $\alpha = \lim \gamma_n$, a $\langle \gamma_n \rangle$-instruction is an instruction p such that for some recursive sequence $e_0, e_1, \ldots$, each e_n is a $\Delta^0_{\gamma_n}$ index for the restriction of p to nodes of level n in the tree and their labels.

Theorem (Ash [2]). *Every Δ^0_α instruction (or $\langle \gamma_n \rangle$-instruction for α a limit ordinal) in an α-system has a correct labelling with an r.e. adherent point.*

9. Main result

We are now in a position to state and prove

Theorem 2. *Let $\mathfrak{A}$ be a recursive structure and R a new relation on $\mathfrak{A}$ satisfying the following conditions:*

(1) *The existential diagram of $\mathfrak{A}$ is recursive, and R is recursive.*

(2) *The relations $\leqslant_\beta$ (for $1 \leqslant \beta < \alpha$) are uniformly recursive.*

(3) *There is a recursive procedure to determine, for each $\boldsymbol{p}$ and x, whether $x \in \mathrm{Rcl}_\alpha(\boldsymbol{p})$.*

Then R is intrinsically Σ^0_α iff it is formally Σ^0_α.

Proof. We have already proved one direction in Lemma 1, so suppose R is not formally Σ^0_α. Then by Lemma 8, we have that for no $\boldsymbol{p}$ is $R = \mathrm{Rcl}_\alpha(\boldsymbol{p})$. So given this, we want to construct a recursive structure $\mathfrak{B}$ together with an isomorphism $f:\mathfrak{B}\cong\mathfrak{A}$ such that $f^{-1}(R)$ is not Σ^0_α.

If $|\mathfrak{A}|$ is finite, then R is trivially formally Σ^0_1, so we suppose $|\mathfrak{A}|=\{a_0, a_1, \ldots\}$ and let $B=\{b_0, b_1, \ldots\}$ be any recursive set. Let L be the language for $\mathfrak{A}$, and let $\{\theta_0, \theta_1, \ldots\}$ be an enumeration of the atomic sentences of $L(B)$.

We obtain a recursive structure $\mathfrak{B}$ with domain B by enumerating during our construction the atomic diagram $D(\mathfrak{B})$ of $\mathfrak{B}$. That is, in the terminology of Section 8, by obtaining an r.e. point m of the metric space $2^{\mathbb{N}}$ with the usual basis, which yields $D(\mathfrak{B})$ by

$$\theta_k \in D(\mathfrak{B}) \quad \text{if } m(k)=1,$$
$$\neg\theta_k \in D(\mathfrak{B}) \quad \text{if } m(k)=0.$$

This will give us a recursive structure, because the basic open sets of $2^{\mathbb{N}}$ are clopen.

So the basic open sets $\sigma\in B(2^{\mathbb{N}})$ correspond to consistent finite subsets $\Sigma(\sigma)$ of $\{\theta_k \mid k\in\mathbb{N}\}\vee\{\neg\theta_k \mid k\in\mathbb{N}\}$.

We will obtain $f:\mathfrak{B}\cong\mathfrak{A}$ as the union of a limiting chain of partial functions. To ensure it is a bijection we ensure that each element of $|\mathfrak{A}|$ and B is used in the construction. To ensure it is an isomorphism we use the notion of coherence defined below. To ensure it is the right isomorphism, we want to satisfy the requirements Req_e: $f^{-1}(R)\neq S_e$ where S_e is the e-th Σ^0_α set.

Let P be the set of all finite one–one partial functions from B to $|\mathfrak{A}|$. $f\in P$ is *coherent w.r.t.* $\sigma\in B(2^{\mathbb{N}})$ if there is a bijection $g:B\to|\mathfrak{A}|$ extending f such that for each $\theta(b_{i_1}, \ldots, b_{i_k})\in\Sigma(\sigma)$, we have $\mathfrak{A}\models\theta[g(b_{i_1}), \ldots, g(b_{i_k})]$. We can obtain an existential formula φ and elements $\boldsymbol{a}\in|\mathfrak{A}|$ such that f is coherent w.r.t. σ iff $\mathfrak{A}\models\varphi[\boldsymbol{a}]$. So by condition (1), this relation between f and σ is recursive.

Definition. For $f, g\in P$ and $\beta\geqslant 1$, define $f\leqslant_\beta g$ if $\mathrm{dom}(f)\subseteq\mathrm{dom}(g)$ and $a_1, \ldots, a_n\leqslant_\beta b_1, \ldots, b_n$ where $\{a_1, \ldots, a_n\}=\mathrm{ran}(f)$ and $b_i=g(f^{-1}(a_i))$.

Lemma 9. *If $f\in P$ is coherent w.r.t. $\sigma\in B(2^{\mathbb{N}})$, then for each $\boldsymbol{a}\in|\mathfrak{A}|$, $\boldsymbol{b}\in B$, $n\in\mathbb{N}$ there exists $g\in P$ and $\sigma'\in B(2^{\mathbb{N}})$ such that $g\supseteq f$, $\boldsymbol{a}\in\mathrm{ran}(g)$, $\boldsymbol{b}\in\mathrm{dom}(g)$, $\sigma'\subseteq\sigma$,* diameter$(\sigma')<1/n$*, and g is coherent w.r.t. σ'.*

Proof. This simply says that we can extend partial functions, maintaining coherence with a finite number of atomic sentences, to include in the domain and range any given elements, and then decide finitely more atomic sentences to maintain the coherence of this partial structure. (If it doesn't force us to choose a particular one of θ_k and $\neg\theta_k$, then we may choose either ad libitum.) □

Lemma 10. *If $f\leqslant_\beta g$ and g is coherent w.r.t. σ, then so is f.*

Proof. The relation $\boldsymbol{a}\leqslant_\beta\boldsymbol{b}$ is equivalent to the fact that $\boldsymbol{b}$ satisfies every Π_β formula satisfied by $\boldsymbol{a}$. So $f\leqslant_\beta g$ implies $f\leqslant_1 g$. Now g is coherent w.r.t. σ, so ran(g) satisfies a certain existential formula. But ran(f) must satisfy the same formula, and so must be coherent w.r.t. σ. $\square$

Lemma 11. *If $\alpha_k > \cdots > \alpha_1 > \alpha_0 \geq 1$ and $f_k \leq_{\alpha_k} f_{k-1} \cdots \leq_{\alpha_2} f_1 \leq_{\alpha_1} f_0$ where f_0 is coherent w.r.t. σ, then there exists $g \supseteq f_k$, coherent w.r.t. σ, such that $f_i \leq_{\alpha_i} g$ for $i = 0, 1, \ldots, k-1$.*

Proof. As in [1]. $\square$

Now, to prove our theorem we define an α-system as follows:

The nodes of the tree of level $n+1$ consist of sequences of the form

$$(\mu_{(0)_1}((0)_2, (0)_3) = k_0, \ldots, \mu_{(n)_1}((n)_2, (n)_3) = k_n)$$

where $i = \langle (i)_1, (i)_2, (i)_3 \rangle$ is a standard tripling function, k_i is 0 or 1, and $\mu_i(m, s) = 1$ (resp. 0) is intended to mean that the i-th Turing machine with Δ^0_α oracle and input m halts (resp. does not halt) within s steps.

A suitable label for a node is a sequence $(f_0, x_0, f_1, x_1, \ldots, f_k, x_k)$ where $f_i \in P$; $f_i \subseteq f_{i+1}$; $x_i \in R - \mathrm{Rcl}_\alpha(\mathrm{ran}(f_i))$ or $x_i = *$; $x_i, a_i \in \mathrm{ran}(f_{i+1})$; and $b_i \in \mathrm{dom}(f_{i+1})$ for each i.

A correct next label at a successor node of level $n+1$ is as follows. If for some j, such that $(j)_1$ is least; $\mu_{(j)_1}((j)_2, (j)_3) = 1$, $(j)_1 = i < k$ and $(j)_2 = f^{-1}_{i+1}(x_i)$ then a correct next label is a suitable label $(f_0, x_0, \ldots, f_i, *, g_{i+1}, x_{i+1})$ such that $g_{i+1} \circ f^{-1}_{i+1}(x_i) \notin R$ and $f_k \leq_{\gamma_n - 1} g_{i+1}$. (If α is a limit $\gamma_n \to \alpha$; otherwise take $\gamma_n = \alpha$ here.) Otherwise, if there is no such j, a correct next label is a suitable label $(f_0, x_0, \ldots, f_k, x_k, f_{k+1}, x_{k+1})$.

The symbol $*$ is intended to show that here Req_i has been satisfied, for we now know that, putting $y = f^{-1}_{i+1}(x_i)$, we have $y \in S_i$ and $g_{i+1}(y) \notin R$.

For each label $l = (f_0, x_0, \ldots, f_k, x_k)$, $F(l)$ is defined to be the set of points of $2^{\mathbb{N}}$ which correspond to structures $\mathfrak{B}$ for which there is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ extending f_k. So $F(l) \cap \sigma \neq \emptyset$ iff f_k is coherent w.r.t. σ.

We define $(f_0, x_0, \ldots, f_k, x_k) \lhd_\beta (g_0, y_0, \ldots, g_l, y_l)$ iff $f_k \leq_\beta g_l$.

Now we must verify that this forms an α-system.

(1) We can take as a label for the root node $\emptyset$, the empty partial function $\emptyset$, together with any $x \in R - \mathrm{Rcl}_\alpha(\emptyset)$.

(2)–(6) are clear from the definitions.

(7) This requires that, given $\sigma \in B(2^{\mathbb{N}})$, $\gamma_n > \alpha_k > \cdots > \alpha_0 \geq 1$ and labels $l_k \lhd_{\alpha_k} \cdots \lhd_{\alpha_2} l_1 \lhd_{\alpha_1} l_0$, where $l_k = (f_0, \ldots, f_m, x_m)$ say and the last function in label l_i is h_i $(0 \leq i \leq k-1)$, such that h_0 is coherent w.r.t. σ, we can find a correct next label l. l can be of two forms, depending on conditions at the node of l_k.

If $l = (f_0, x_0, \ldots, f_m, x_m, f_{m+1}, x_{m+1})$, a suitable coherent f_{m+1} exists by Lemma 11, since we have $f_m \lhd_{\alpha_k} \cdots \lhd_{\alpha_2} h_1 \lhd_{\alpha_1} h_0$ and require $h_i \lhd_{\alpha_i} f_{m+1}$ (for $0 \leq i \leq k-1$). By Lemma 9, we may take $b_m \in \mathrm{dom}(f_{m+1})$ and $a_m, x_m \in \mathrm{ran}(f_{m+1})$. An appropriate x_{m+1} exists because $R \neq \mathrm{Rcl}_\alpha(\mathrm{ran}(f_{m+1}))$, and can be found by condition (3) of the theorem.

In the other case, suppose i is the least for which the conditions involving $f_{i+1}^{-1}(x_i)$ are satisfied. Then we have to find a coherent g_{i+1} such that $g_{i+1} \circ f_{i+1}^{-1}(x_i) \notin R$ and $f_m \leqslant_{\gamma_{n-1}} g_{i+1}$. We have $f_i \leqslant_{\gamma_n} f_m \leqslant_{\alpha_k} h_{k-1} \cdots \leqslant_{\alpha_2} h_1 \leqslant_{\alpha_1} h_0$. (Note that if $f \subseteq g$, then $f \leqslant_\beta g$ for every β.) By Lemma 11, there is a coherent $g \supseteq f_i$ such that $h_j \leqslant_{\alpha_j} g$ (for $j = 0, \ldots k-1$) and $f_m \leqslant_{\gamma_{n-1}} g$.

By Lemma 9, there is a coherent $g' \supseteq g$ such that $x_i \in \operatorname{ran}(g')$.

Now put $\operatorname{ran}(f_i) = \boldsymbol{a}$, and $\operatorname{ran}(g') = \boldsymbol{a}, x_i, \boldsymbol{b}$. Since $x_i \notin \operatorname{Rcl}_\alpha(\boldsymbol{a})$ there exist $x', \boldsymbol{b}'$ such that $\boldsymbol{a}, x_i, \boldsymbol{b} \leqslant_{\gamma_n - 1} \boldsymbol{a}, x', \boldsymbol{b}'$ and $x' \notin R$. So, letting $k : x_i \to x', \boldsymbol{b} \to \boldsymbol{b}'$, put $g'' = k \circ g'$.

By Lemma 9, there is a coherent $g_{i+1} \supseteq g''$ with $b_i \in \operatorname{dom}(g_{i+1})$ and $a_i \in \operatorname{ran}(g_{i+1})$.

We have $g_{i+1} \supseteq g'' \supseteq f_i$ and $g_{i+1} \supseteq g'' \geqslant_{\gamma_n - 1} g' \supseteq g \geqslant_{\alpha_j} h_j$ (for $j = 0, \ldots, k-1$) as required.

Also $g_{i+1} \geqslant_{\gamma_n - 1} g \geqslant_{\gamma_n - 1} f_m$, so $(f_0, x_0, \ldots, f_i, *, g_{i+1}, x'_{i+1})$ is a correct next label.

Thus we have defined an α-system.

Furthermore, the path through

$$(\mu_{(0)_1}((0)_2, (0)_3) = k_0, \ldots, \mu_{(n)_1}((n)_2, (n)_3) = k_n)$$

where $\mu_i(m, s) = 1$ (resp. 0) if the i-th Turing machine with Δ^0_α oracle and input m actually does (resp. does not) halt within s steps, is generated by a Δ^0_α instruction. (In this case, the label on a node is irrelevant in determining the appropriate successor node.)

In the limit ordinal case, we can take it as a $\langle \gamma_n \rangle$-instruction by ignoring all calls to oracles higher than $\Delta^0_{\gamma_n}$ at nodes of level n.

So this path has a correct labelling (f_0^0, x_0^0), $(f_0^1, x_0^1, f_1^1, x_1^1), \ldots, (f_0^n, x_0^n, \ldots, f_n^n, x_n^n), \ldots$ with an r.e. adherent point. This point defines the recursive structure $\mathfrak{B}$. It remains only to construct the isomorphism $f : \mathfrak{B} \cong \mathfrak{A}$ from the labels.

Lemma 12. *For each n, $\lim_i f_n^i = f_n$ and $\lim_i x_n^i = x_n$ exist and the f_n form a chain.*

Proof. This is a standard finite injury argument. Nothing can cause f_0^i to change. The only way x_0^i can change is if at some label it turns into an $*$; in this case it remains thus thereafter. f_1^i can be changed only if x_0^i becomes $*$. So each f_n^i and x_n^i can only be changed finitely often and so must all settle down. The f_n form a chain simply because the f_n^i do for each i. $\square$

Lemma 13. *Each requirement* Req_n *is satisfied.*

Proof. If $x_n = *$, then as noted before, for some y, $f(y) = f_{n+1}(y) \notin R$ but $y \in S_n$.

If x_n is not an $*$, then we must have $f^{-1}(x_n) = f_{n+1}^{-1}(x_n) \notin S_n$. But $x_n \in R$. $\square$

So we have constructed a recursive structure $\mathfrak{B}$ and an isomorphism $f:\mathfrak{B}\cong\mathfrak{A}$ such that $f^{-1}(R)$ is not Σ^0_α.

10. A related notion

Let $\hat{\Sigma}^0_\alpha=\bigcup_{\beta<\alpha}\Sigma^0_\beta$ for limit ordinals α.

Theorem 3. *Suppose* $\mathrm{Rcl}_\beta(\boldsymbol{p})\neq R$ *for each* $\beta<\alpha$ *and each* $\boldsymbol{p}$. *Then* R *is no intrinsically* $\hat{\Sigma}^0_\alpha$ *if the following are satisfied.*

(1) *The existential diagram of* $\mathfrak{A}$ *is recursive and* R *is recursive.*

(2) *The relations* $\leqslant_\beta$ *(for* $1\leqslant\beta<\alpha$*) are uniformly recursive.*

(3) *There is a recursive procedure to determine for each* $\boldsymbol{p}$, x *and* $\beta<\alpha$ *whethe* $x\in\mathrm{Rcl}_\beta(\boldsymbol{p})$.

Proof. To prove Theorem 3 we construct an α-system as follows.

The nodes of the tree of level $n+1$ are sequences

$$(\mu^{(0)_1}_{(0)_2}(m_0)=k_0,\ \ldots,\ \mu^{(n)_1}_{(n)_2}(m_n)=k_n)$$

where $\mu^n_e(m)=1$ (resp. 0) is intended to mean that m is (resp. is not) an elemen of the e-th $\Sigma^0_{\gamma_n-1}$ set. We insist that $(i)_1<i$ for each i.

Suitable labels of nodes of level n are of the form (f, x, g) where $f, g\in P$, $f\subseteq g$ $x\in R-\mathrm{Rcl}_{\gamma_n}(\mathrm{ran}(f))$, $a_n, x\in\mathrm{ran}(g)$, and $b_n\in\mathrm{dom}(g)$.

A correct next label on the node

$$(\mu^{(0)_1}_{(0)_2}(m_0)=k_0,\ \ldots,\ \mu^{(n)_1}_{(n)_2}(m_n)=k_n)$$

after the label (f, x, g) is as follows. If $m_n=g^{-1}(x)$ and $k_n=1$, a correct nex label is a suitable label (g', y, h) such that $g'\circ g^{-1}(x)\notin R$, $g'\supseteq f$ and $g\leqslant_{\gamma_n-1}g'$ Otherwise a correct next label is a suitable label (g, y, h).

$F(f, x, g)$ is defined to be the set of points of $2^{\mathbb{N}}$ which correspond to structure $\mathfrak{B}$ for which there is an isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ extending g.

$(f, x, g)\lhd_\beta(h, y, k)$ is defined to mean $g\leqslant_\beta k$.

It can easily be checked, entirely similarly to above that this forms ar α-system.

The desired $\langle\gamma_n\rangle$-instruction is that which proceeds from a node of level n witl label (f, x, g) to the successor node

$$(\mu^{(0)_1}_{(0)_2}(m_0)=k_0,\ \ldots,\ \mu^{(n)_1}_{(n)_2}(\overset{g^{-1}(x)}{\cancel{x}})=k)$$

where

$$k=\begin{cases}0 & \text{if } \overset{g^{-1}(x)}{\cancel{x}}\notin(n)_2\text{-th }\Sigma^0_{\gamma_{(n)_1}-1}\text{ set},\\ 1 & \text{otherwise.}\end{cases}$$

This is a $\langle\gamma_n\rangle$-instruction because $(n)_1<n$, and so has an r.e. adherent point which determines a structure $\mathfrak{B}$, and a correct labelling (f_0, x_0, g_0) $(f_1, x_1, g_1),\ldots.$ Clearly $\{f_i\}$ form a chain, and $\bigcup f_i:\mathfrak{B}\cong\mathfrak{A}$.

Furthermore, we see that at level n in this chain we have ensured that $f^{-1}(R) \neq (n)_2$-th $\Sigma^0_{\gamma(n)_1 - 1}$ set. Since $\lim(\gamma_n - 1) = \alpha$, $f^{-1}(R) \notin \hat{\Sigma}^0_\alpha$, as required. □

Corollary. *Under the assumptions of Theorem* 3, *a relation R is intrinsically $\hat{\Sigma}^0_\alpha$ on a recursive structure $\mathfrak{A}$ iff it is intrinsically Σ^0_β on $\mathfrak{A}$ for some $\beta < \alpha$.*

11. Example

For an example of the application of these ideas, we consider recursive well-orderings $(\sigma, <)$ for constructive ordinals σ.

Lemma 14. *In any recursive well-ordering $(\sigma, <)$ the set L_α of α*-th *limit points is formally $\Pi^0_{2\cdot\alpha}$.*

Proof. There are simple $\Pi_{2\cdot\alpha}$ formulae defining the L_α:

$$x \in L_1 \Leftrightarrow \forall y\, \exists z\, [y < x \rightarrow y < z < x],$$
$$x \in L_{\alpha+1} \Leftrightarrow \forall y\, \exists z\, [(y < x \;\&\; y \in L_\alpha) \rightarrow (y < z < x \;\&\; z \in L_\alpha)].$$

If $\alpha = \lim \gamma_n$, then $x \in L_\alpha \Leftrightarrow \bigwedge\!\!\!\bigwedge_n x \in L_{\gamma_n}$. □

Clearly if L_α is empty or finite, then it is also formally $\Sigma^0_{2\cdot\alpha}$. (Of course, it is formally Σ^0_1, as it has the definition

$$x \in L_\alpha \Leftrightarrow \bigvee_{l_i \in L_\alpha} x = l_i$$

which involves only the finitely many parameters l_i.) But we will see that if L_α is infinite, then it is not intrinsically $\Sigma^0_{2\cdot\alpha}$.

To apply Theorem 2, we have to consider the relations $\leqslant_\gamma$.

Lemma 15 (Ash [2]). *The relations $\boldsymbol{a} \leqslant_\gamma \boldsymbol{b}$ for recursive well-orderings are equivalent to certain conditions on the order types of the intervals between the a_i and the intervals between the b_i. These conditions are such that given constructive ordinals σ, α, there exists a recursive copy of σ for which the $\leqslant_\alpha (\gamma < \alpha)$ are uniformly recursive.*

To fulfil condition (3), it suffices to be able to find, for any $\boldsymbol{a}$, an element of $R - \mathrm{Rcl}_{2\cdot\alpha}(\boldsymbol{a})$.

Lemma 16. *Suppose $(\sigma, <)$ is a recursive well-ordering such that $\sigma \geqslant \omega^{\alpha+1}$. Let $R = L_\alpha$, the set of α*-th *limit points. Then, given any $\boldsymbol{a}$,*

$$x = \max_{a_i \in \text{initial } \omega^{\alpha+1}} \{a_i\} + \omega^\alpha \notin \mathrm{Rcl}_{2\cdot\alpha}(\boldsymbol{a}).$$

Proof. Suppose α is a successor ordinal. Let $\boldsymbol{b}$ be given. We must be able to find x', $\boldsymbol{b}'$ such that $\boldsymbol{a}, x, \boldsymbol{b} \leqslant_{2\cdot\alpha-1} \boldsymbol{a}, x', \boldsymbol{b}'$ and $x' \notin L_\alpha$.

It may be checked that $x' = x + \omega^{\alpha-1}$ and

$$b_i' = \begin{cases} b_i + \omega^{\alpha-1} & \text{if } x \leqslant b_i < x + \omega^\alpha, \\ b_i & \text{otherwise} \end{cases}$$

are suitable, using the conditions of Lemma 15.

Now suppose α is a limit ordinal (so $2 \cdot \alpha = \alpha$). Let $\beta < \alpha$ and $\boldsymbol{b}$ be given. We want to find x', $\boldsymbol{b}'$ such that $\boldsymbol{a}, x, \boldsymbol{b} \leqslant_\beta \boldsymbol{a}, x', \boldsymbol{b}'$ and $x' \notin L_\alpha$. Again it may be checked that $x' = x + \omega^\beta$, and

$$b_i' = \begin{cases} b_i + \omega^{\beta} & \text{if } x \leqslant b_i < x + \omega^{\beta+1}, \\ b_i & \text{otherwise} \end{cases}$$

are suitable. $\square$

Hence we have

Example. *If $(\sigma, <)$ is a recursive well-ordering such that $\sigma \geqslant \omega^{\alpha+1}$, then L_α, the set of α*-th *limit points of σ, is not intrinsically $\Sigma^0_{2\cdot\alpha}$.*

Proof. Take a sufficiently recursive copy of $(\sigma, <)$. Specifically, it will suffice if the function $(a, b) \mapsto$ the order type of $[a, b]$, for $a < b$, is effective. For then the existential diagram of $(\sigma, <)$, and L_α are recursive. And by Lemmas 15 and 16, conditions (2) and (3) of Theorem 2 are satisfied. $\square$

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